Holomorphic curves - solutions to exercises

Miguel Pereira

2018-2019 Summer Semester

Contents

1 Exercise sheet No. 1 - 25-04-2019

A **symplectic manifold** is a pair (M, ω) where M is a differentiable manifold and $\omega \in \Omega^2(M)$ is a 2-form that is closed and nondegenerate.

The following lemma will be used in the solution of exercises 1.1 and 1.2.

Lemma 1. *Let (V, ω) be a finite dimensional symplectic vector space (i.e. ω is a 2-form on V that is nondegenerate). Then there exists a basis of V, $(u_1, \dots, u_n, v_1, \dots, v_n)$ such that if $(u^1, \dots, u^n, v^1, \dots, v^n)$ is the associated dual basis of V^*, then $\omega = \sum_{i=1}^n u^i \wedge v^i$.*

Proof:

1. There exist $u_1, v_1 \in V$ such that $\omega(u_1, v_1) = 1$ and u_1, v_1 are linearly independent.

   Proof:

   1.1. Let $u_1 \in V \backslash \{0\}$.

   1.2. There exists $v_1 \in V$ such that $\omega(u_1, v_1) = 1$.

   Proof: Since $u_1 \neq 0$, ω is nondegenerate, and we can make $\omega(u_1, v_1) = 1$ by rescaling.

   1.3. u_1, v_1 are linearly independent.

   Proof: If u_1, v_1 were linearly dependent, $\omega(u_1, v_1) = 0$.

2. If $u_1, \dots, u_m, v_1, \dots, v_m \in V$ are such that

   - $\forall i, j = 1, \dots, m\colon\ \omega(u_i, u_j) = 0$, $\omega(v_i, v_j) = 0$, $\omega(u_i, v_j) = \delta_{ij}$,
   - $u_1, \dots, u_m, v_1, \dots, v_m$ is a linearly independent set,
   - $\dim V > 2m$,

   then there exist u_{m+1}, v_{m+1} such that

   - $\forall i, j = 1, \dots, m+1\colon\ \omega(u_i, u_j) = 0$, $\omega(v_i, v_j) = 0$, $\omega(u_i, v_j) = \delta_{ij}$,
   - $u_1, \dots, u_{m+1}, v_1, \dots, v_{m+1}$ is a linearly independent set.

   Proof:

   2.1. Let

   $$W := \operatorname{span}\{u_1, \dots, u_m, v_1, \dots, v_m\}$$
   $$W^\omega := \{u \in V \mid \forall v \in W\colon \omega(u, v) = 0\}.$$

   2.2. $V = W \oplus W^\omega$.

   Proof:

   2.2.1. $V = W + W^\omega$.

   Proof:

   2.2.1.1. It suffices to assume that $v \in V$ and prove that there exist $w \in W$, $w^\omega \in W^\omega$ such that $v = w + w^\omega$.

   2.2.1.2. Let $w = \sum_{i=1}^m \omega(v, v_i) u_i - \omega(v, u_i) v_i \in W$.

   2.2.1.3. Let $w^\omega = v - w$. Then $w^\omega \in W^\omega$.

   Proof: It suffices to assume that $u \in W$, and prove that $\omega(w^\omega, u) = 0$. There exist $a_1, \dots, a_m, b_1, \dots, b_m \in \mathbb{R}$ such that

   $$u = \sum_{i=1}^m a_i u_i + b_i v_i.$$

$$
\begin{aligned}
\omega(w^\omega, u) &= \omega(v - w, u) \\
&= \omega(v, u) - \omega(w, u) \\
&= 0,
\end{aligned}
$$

where the last step follows from expanding w, u in a basis and using equations $\omega(u_i, u_j) = 0$, $\omega(v_i, v_j) = 0$, $\omega(u_i, v_j) = \delta_{ij}$.

2.2.2. $W \cap W^\omega = \{0\}$.

Proof: It suffices to assume that $u \in W$, $u \in W^\omega$ and show that $u = 0$. Since $u \in W$, $\forall v \in W^\omega\colon \omega(u, v) = 0$. Since $u \in W^\omega$, $\forall v \in W\colon \omega(u, v) = 0$. The result follows from step 2.2.1 and nondegeneracy of ω.

2.3. W^ω is nonempty. Let $u_{m+1} \in W^\omega$.

Proof: $\dim V > 2m$ implies that $\dim W^\omega > 0$.

2.4. There exists $v_{m+1} \in W^\omega$ such that $\omega(u_{m+1}, v_{m+1}) = 1$.

Proof:

2.4.1. There exists $v \in V$ such that $\omega(u_{m+1}, v) = 1$.

Proof: ω is nondegenerate and $u_{m+1} \neq 0$.

2.4.2. There exist $w \in W$, $w^\omega \in W^\omega$ such that $v = w + w^\omega$.

Proof: Step 2.2.

2.4.3. $v_{m+1} = w^\omega$ is as desired.

Proof:

$$
\begin{aligned}
1 &= \omega(u_{m+1}, v) \\
&= \omega(u_{m+1}, w) + \omega(u_{m+1}, w^\omega) \\
&= \omega(u_{m+1}, w^\omega).
\end{aligned}
$$

2.5. $u_1, \ldots, u_{m+1}, v_1, \ldots, v_{m+1}$ is a linearly independent set.

Proof: $u_1, \ldots, u_m, v_1, \ldots, v_m$ is a basis of W, u_{m+1}, v_{m+1} is a linearly independent set in W^ω, and $V = W \oplus W^\omega$.

3. Q.E.D.

Proof: Start with u_1, v_1 from step 1. Apply step 2 many times until its no longer true that $\dim V > 2m$ (this eventually happens because V is finite dimensional). The set that we end up with is the desired basis. □

Exercise 1.1. Show that every symplectic manifold is even dimensional.

Solution: Let $p \in M$. $\dim M = \dim T_pM$, (T_pM, ω_p) is a symplectic vector space, and Lemma 1. □

Exercise 1.2. Show that if (M, ω) is a $2n$-dimensional symplectic manifold, then ω^n is a volume form on M.

Solution:

1. It suffices to assume that $p \in M$, and prove that $\omega^n|_p \neq 0$.
2. There exists a basis $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ of T_pM such that $\omega_p = \sum_{i=1}^m u^i \wedge v^i$, where $(u^1, \ldots, u^n, v^1, \ldots, v^n)$ is the basis of T_p^*M dual to $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ of T_pM.

Proof: (T_pM, ω_p) is a symplectic vector space, and Lemma 1.

3. $\omega^n|_p = n!u^1 \wedge v^1 \wedge \cdots \wedge u^n \wedge v^n$.

   Proof:

$$\begin{aligned}\omega^n|_p &= \omega_p \wedge \cdots \wedge \omega_p \\ &= (u^1 \wedge v^1 + \cdots + u^n \wedge v^n) \\ &\quad \wedge (u^1 \wedge v^1 + \cdots + u^n \wedge v^n) \\ &\quad \vdots \\ &\quad \wedge (u^1 \wedge v^1 + \cdots + u^n \wedge v^n) \\ &= n!u^1 \wedge v^1 \wedge \cdots \wedge u^n \wedge v^n.\end{aligned}$$

4. Q.E.D.

   Proof: $u^1 \wedge v^1 \wedge \cdots \wedge u^n \wedge v^n \neq 0$. □

Exercise 1.3. Let N be a differentiable manifold and consider its cotangent bundle $\pi\colon T^*N \longrightarrow N$. Consider the 1-form λ on T^*N that is given as follows: at $e \in T^*N$, $\lambda_e = \pi^*e$. Show that $\omega = d\lambda$ is a symplectic form on T^*N.

Solution:

1. ω is closed.

   Proof: ω is exact.

2. ω is nondegenerate.

   Proof:

   2.1. For each coordinate neighborhood $(U, q_1, \dots, q_n)$ in N, define functions
   $$p_i\colon T^*U \longrightarrow \mathbb{R}$$
   as follows. If $e \in T^*U \subset T^*N$, $e = \sum_{i=1}^n p_i(e) dq_i|_{\pi(e)}$. Then,
   $$(T^*U, q_1, \dots, q_n, p_1, \dots, p_n)$$
   are coordinate neighborhoods in T^*N and all such neighborhoods form an atlas for T^*N.

   2.2. In canonical coordinates of T^*N, λ is given by $\lambda = \sum_{i=1}^n p_i dq_i$.

   Proof: It suffices to assume that $e \in T^*N$ and prove that $\lambda = \sum_{i=1}^n p_i(e) dq_i|_e$.
   $$\begin{aligned}\lambda_e &= \pi^*e \\ &= \pi^* \sum_{i=1}^n p_i(e) dq_i|_{\pi(e)} \\ &= \sum_{i=1}^n p_i(e) d(q_i \circ \pi)|_e \\ &= \sum_{i=1}^n p_i(e) dq_i|_e.\end{aligned}$$

   2.3. In canonical coordinates of T^*N, ω is given by $\omega = \sum_{i=1}^n dp_i \wedge dq_i$.

   Proof:
   $$\begin{aligned}\omega &= d\lambda \\ &= d\sum_{i=1}^n p_i dq_i\end{aligned}$$

$$= \sum_{i=1}^{n} dp_i \wedge dq_i.$$

2.4. Q.E.D.

Proof: It suffices to assume that $\xi = \sum_{i=1}^n u_i \frac{\partial}{\partial q_i} + v_i \frac{\partial}{\partial p_i} \in T_e(T^*N)$ is such that $\omega(\xi, \cdot) = 0$, and prove that $\xi = 0$.

$$\begin{aligned} 0 &= \omega(\xi, \cdot) \\ &= \sum_{i=1}^{n} dp_i(\xi) dq_i - dq_i(\xi) dp_i \\ &= \sum_{i=1}^{n} v_i dq_i - u_i dp_i. \end{aligned}$$

Since the dp_i, dq_i are linearly independent, this implies that all the u_i, v_i are 0, and so $\xi = 0$. $\square$

Let $f \in C^\infty(M, \mathbb{R})$. Since ω is nondegenerate, there exists a unique vector field X_f in M such that $\omega(X_f, \cdot) = -df$. This vector field is called the **Hamiltonian vector field** of f. Let $\phi_f^t := \phi_{X_f}^t : M \longrightarrow M$ denote the flow of X_f.

Exercise 1.4. Show that for all $x \in M$, $f(\phi_f^t(x))$ is independent of t.

Solution: It suffices to assume that $x \in M$ and prove that for all t

$$\frac{d}{dt} f(\phi_f^t(x)) = 0.$$

$$\begin{aligned} \frac{d}{dt} f(\phi_f^t(x)) &= df(\phi_f^t(x)) \frac{d}{dt} \phi_f^t(x) && \text{[chain rule]} \\ &= df(\phi_f^t(x)) X_f(\phi_f^t(x)) && \text{[def. of flow of a vector field]} \\ &= \omega(X_f(\phi_f^t(x)), X_f(\phi_f^t(x))) && \text{[def. of Hamiltonian vector field]} \\ &= 0 && [\omega \text{ is a form}]. \end{aligned}$$ $\square$

Exercise 1.5. Show that

(i) $L_{X_f} \omega = 0$;

(ii) $(\phi_f^t)^* \omega = \omega$ for all t.

Solution:

1. (i)

   Proof:

$$\begin{aligned} L_{X_f} \omega &= \iota_{X_f} d\omega + d\iota_{X_f} \omega && \text{[Cartan's magic formula]} \\ &= d\iota_{X_f} \omega && [\omega \text{ is closed}] \\ &= -d^2 f && [\text{definition of } X_f] \\ &= 0. \end{aligned}$$

2. (ii)

   Proof:

   2.1. It suffices to show that for all t,

$$\frac{d}{dt} (\phi_f^t)^* \omega = 0.$$

Proof: If $\frac{d}{dt}(\phi_f^t)^*\omega = 0$, then $(\phi_f^t)^*\omega$ does not depend on t and

$$(\phi_f^t)^*\omega = (\phi_f^0)^*\omega = \mathrm{id}_M^*\,\omega = \omega.$$

2.2. Q.E.D.

Proof:

$$\begin{aligned}\frac{d}{dt}(\phi_f^t)^*\omega &= (\phi_f^t)^* L_{X_f}\omega && \text{[def. of Lie derivative and properties of the flow]}\\ &= (\phi_f^t)^* 0\\ &= 0.\end{aligned}$$

□

Let $f, g \in C^\infty(M,\mathbb{R})$. The **Poisson bracket** of f and g, denoted $\{f, g\}$, is a function on M defined by $\{f, g\} = \omega(X_f, X_g)$.

The following lemma will be used in the solution of exercises 1.6 and 1.7.

Lemma 2. *Let $f, g \in C^\infty(M,\mathbb{R})$. Then,*

(i) $\{f, g\} = X_f \cdot g$;

(ii) $[X_f, X_g] = X_{\{f,g\}}$.

Proof:

1. $\{f, g\} = X_f \cdot g$.

Proof:

$$\begin{aligned}\{f, g\} &= \omega(X_f, X_g)\\ &= -\omega(X_g, X_f)\\ &= dg(X_f)\\ &= X_f \cdot g.\end{aligned}$$

2. $[X_f, X_g] = X_{\{f,g\}}$.

Proof: By nondegeneracy of ω, it suffices to show that $\omega([X_f, X_g], \cdot) = \omega(X_{\{f,g\}}, \cdot)$.

$$\begin{aligned}\omega(X_{\{f,g\}}, \cdot) &= -d(\{f, g\}) && \text{[def. of Hamiltonian vector field]}\\ &= -d(X_f \cdot g) && \text{[Step 1]}\\ &= -d(dg(X_f))\\ &= -d\iota_{X_f} dg\\ &= -(d\iota_{X_f} + \iota_{X_f} d)dg && [d^2 = 0]\\ &= L_{X_f}\iota_{X_g}\omega && \text{[Cartan's magic formula and def. of } X_g]\\ &= \iota_{L_{X_f}X_g}\omega - \iota_{X_g} L_{X_f}\omega && \text{[Lie derivative of forms formula]}\\ &= \omega([X_f, X_g], \cdot) && \text{[Exercise 1.5 (i)].}\end{aligned}$$

□

Exercise 1.6. Show that $C^\infty(M,\mathbb{R})$ equipped with the Poisson bracket $\{\cdot,\cdot\}$ is a Lie algebra.

Solution:

1. $\{\cdot,\cdot\}$ is antisymmetric.

Proof: It suffices to assume that $f, g \in C^\infty(M,\mathbb{R})$ and prove that $\{f, g\} = \{g, f\}$.

$$\begin{aligned}\{f, g\} &= \omega(X_f, X_g)\\ &= -\omega(X_g, X_f)\\ &= -\{g, f\}.\end{aligned}$$

2. $\{\cdot,\cdot\}$ is bilinear.

   Proof: By antisymmetry, it suffices to assume that $a, b \in \mathbb{R}$, $f, g, h \in C^\infty(M, \mathbb{R})$ and prove that
$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}.$$
$$\begin{aligned}
\{af + bg, h\} &= \omega(X_{af+bg}, X_h) \\
&= \omega(aX_f + bX_g, X_h) && [\text{The map } f \longmapsto X_f \text{ is linear}] \\
&= a\omega(X_f, X_h) + b\omega(X_g, X_h) \\
&= a\{f, h\} + b\{g, h\}.
\end{aligned}$$

3. $\{\cdot,\cdot\}$ satisfies the Jacobi identity.

   Proof: It suffices to assume that $f, g, h \in C^\infty(M, \mathbb{R})$, and prove that
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$
$$\begin{aligned}
&\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \\
&\quad = \{f, \{g, h\}\} - \{g, \{f, h\}\} - \{\{f, g\}, h\} && [\text{antisymmetry}] \\
&\quad = X_f \cdot (X_g \cdot h) - X_g \cdot (X_f \cdot h) - X_{\{f,g\}} \cdot h && [\text{Lemma 2 (i)}] \\
&\quad = \Big([X_f, X_g] - X_{\{f,g\}}\Big) \cdot h \\
&\quad = 0 && [\text{Lemma 2 (ii)}].
\end{aligned}$$
$\square$

Exercise 1.7. Show that the map
$$\begin{aligned}
C^\infty(M, \mathbb{R}) &\longrightarrow \mathfrak{X}(M) \\
f &\longmapsto X_f
\end{aligned}$$
is a Lie algebra homomorphism.

Solution: The map is linear and by Lemma 2 it preserves the bracket. $\square$

2 Exercise sheet No. 2 - 02-05-2019

In these exercises we show that the space of almost complex structures that are compatible with a symplectic form is contractible. Our proof is based on the one presented in [MS17].

Let (V,ω) be a symplectic vector space. Let $\mathfrak{Met}(V)$ denote the set of inner products on V. Define

$$\begin{aligned}\tilde{\omega}\colon V &\longrightarrow V^*\\ u &\longmapsto \omega(u,\cdot),\end{aligned}$$

and for each $g \in \mathfrak{Met}(V)$ define

$$\begin{aligned}\tilde{g}\colon V &\longrightarrow V^*\\ u &\longmapsto g(u,\cdot).\end{aligned}$$

For each $J \in \mathcal{J}(V,\omega)$ let $g_J \coloneqq \omega(\cdot, J\cdot)$.

Define a map

$$\begin{aligned}A\colon \mathfrak{Met}(V) &\longrightarrow \mathrm{Aut}(V)\\ g &\longmapsto A_g \coloneqq (\tilde{g})^{-1}\circ\tilde{\omega}.\end{aligned}$$

Lemma 3.

(i) $\forall g \in \mathfrak{Met}(V)$: *the g-adjoint of A_g is $-A_g$;*

(ii) $\forall g \in \mathfrak{Met}(V)$: $P_g \coloneqq A_g^* A_g$ *is g-self adjoint and g-positive definite.*

Proof:

1. (i)

   Proof: It suffices to assume that $u, v \in V$ and prove that $g(A_g u, v) = g(u, A_g v)$.

$$\begin{aligned} g(A_g u, v) &= \tilde{g}\circ A_g(u)(v) && [\text{Def. of } \tilde{g}]\\ &= \tilde{\omega}(u)(v) && [\text{Def. of } A_g]\\ &= -\tilde{\omega}(v)(u) && [\omega \text{ is antisymmetric}]\\ &= -\tilde{g}\circ A_g(v)(u) && [\text{Def. of } A_g]\\ &= -g(A_g v, u) && [\text{Def. of } \tilde{g}]\\ &= g(u, -A_g v) && [g \text{ is symmetric}].\end{aligned}$$

2. (ii)

   Proof:

   2.1. It suffices to assume that $u, v \in V$ and prove that $g(P_g u, v) = g(u, P_g v)$ and $g(P_g u, u) \geq 0$.

   2.2. $g(P_g u, v) = g(u, P_g v)$.

   Proof:

$$\begin{aligned} g(P_g u, v) &= g(-A_g^2 u, v) && [\text{Def. of } P_g]\\ &= g(A_g u, A_g v) && [\text{(i)}]\\ &= g(u, -A_g^2 v) && [\text{(i)}]\\ &= g(u, P_g v) && [\text{Def. of } P_g].\end{aligned}$$

2.3. $g(P_g u, u) \geq 0$.

Proof:

$$\begin{aligned} g(P_g u, u) &= g(-A_g^2 u, u) && \text{[Def. of } P_g] \\ &= g(A_g u, A_g u) && \text{[(i)]} \\ &\geq 0 && [g \text{ is positive definite}]. \end{aligned}$$

□

Exercise 2.1. Show that there exists a unique map

$$\begin{aligned} Q \colon \mathfrak{Met}(V) &\longrightarrow \mathrm{Aut}(V) \\ g &\longmapsto Q_g \end{aligned}$$

such that for all $g \in \mathfrak{Met}(V)$

(i) Q_g is g-self adjoint: $\forall u, v \in V \colon g(Q_g u, v) = g(u, Q_g v)$;

(ii) Q_g is g-positive definite: $\forall u \in V \colon g(Q_g u, u) \geq 0$;

(iii) $Q_g^2 = -A_g^2$.

Solution:

1. Uniqueness.

   Proof: P_g is g-positive definite, Q_g is g-positive definite, and $Q_g^2 = P_g$, so Q_g is the g-square root of P_g.

2. Existence.

   Proof:

   2.1. There exists a g-orthonormal basis of V, $\{v_1, \ldots, v_{2n}\}$, such that with respect to this basis $P_g = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n})$, where $\forall i = 1, \ldots, 2n \colon \lambda_i > 0$.

   Proof:

   2.1.1. There exists a g-orthonormal basis of V, $\{v_1, \ldots, v_{2n}\}$, such that with respect to this basis $P_g = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n})$, where $\forall i = 1, \ldots, 2n \colon \lambda_i \in \mathbb{R}$.

   Proof: P is g-self adjoint and the spectral theorem.

   2.1.2. $\forall i = 1, \ldots, 2n \colon \lambda_i > 0$.

   Proof: P_g is g-positive definite.

   2.2. Let Q_g be given in the basis $\{v_1, \ldots, v_{2n}\}$ by $Q_g = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})$.

   2.3. (i)

   Proof: Because of the matrix representation of Q_g.

   2.4. (ii)

   Proof: Because of the matrix representation of Q_g.

   2.5. (iii)

   Proof: $Q_g^2 = P_g = -A_g^2$.

□

Exercise 2.2. Show that

(i) $Q_g \circ A_g = A_g \circ Q_g$;

(ii) $J_g \coloneqq Q_g^{-1} A_g \in \mathcal{J}(V, g) \cap \mathcal{J}(V, \omega)$;

(iii) If g is ω-compatible then $g = g_{J_g}$ and $J_g = (\tilde{g})^{-1} \circ \tilde{\omega}$;

(iv) If J is ω-compatible then $J = J_{g_J}$.

Solution:

1. There exists a g-orthonormal basis of V, $\{v_1, \ldots, v_{2n}\}$, such that with respect to this basis $P_g = \operatorname{diag}(\lambda_1, \ldots, \lambda_{2n})$, $Q_g = \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})$, where $\forall i = 1, \ldots, 2n$: $\lambda_i > 0$.

2. (i)

   Proof:

   2.1. $(QA)_{ij} = \sqrt{\lambda_i} A_{ij}$, $(AQ)_{ij} = A_{ij}\sqrt{\lambda_j}$, $(Q^2 A)_{ij} = \lambda_i A_{ij}$, $(AQ^2)_{ij} = A_{ij}\lambda_j$.

   Proof: $Q_{ij} = \lambda_i \delta_{ij}$.

   2.2. $\lambda_i A_{ij} = A_{ij}\lambda_j$.

   Proof: $Q^2 A = -A^3 = AQ^2$ and the equations from step 2.1.

   2.3. It suffices to show that $\sqrt{\lambda_i} A_{ij} = A_{ij}\sqrt{\lambda_j}$.

   Proof: Equations from step 2.1.

   2.4. $\sqrt{\lambda_i} A_{ij} = A_{ij}\sqrt{\lambda_j}$.

   Proof:

   2.4.1. Case $A_{ij} = 0$.

   Proof: $\sqrt{\lambda_i} A_{ij} = 0 = A_{ij}\sqrt{\lambda_j}$.

   2.4.2. Case $A_{ij} \neq 0$.

   Proof:

$$\begin{aligned} \lambda_i A_{ij} = A_{ij}\lambda_j &\implies \lambda_i = \lambda_j \\ &\implies \sqrt{\lambda_i} A_{ij} = A_{ij}\sqrt{\lambda_j}. \end{aligned}$$

3. (ii)

   Proof:

   3.1. $J_g \coloneqq Q_g^{-1} A_g$ is an almost complex structure on V.

   Proof:

$$\begin{aligned} J_g^2 &= (Q_g^{-1} A_g)^2 && \text{[Def. of } J_g\text{]} \\ &= Q_g^{-1} A_g Q_g^{-1} A_g \\ &= Q_g^{-1} Q_g^{-1} A_g^2 && \text{[Exercise 2.1 (i)]} \\ &= -Q_g^{-1} Q_g^{-1} Q_g^2 && \text{[Exercise 2.1 (iii)]} \\ &= -\operatorname{id} \end{aligned}$$

   3.2. $J_g \in \mathcal{J}(V, g)$.

   Proof:

$$\begin{aligned} J^* g &= g(J\cdot, J\cdot) \\ &= g(Q_g^{-1} A_g\cdot, Q_g^{-1} A_g\cdot) && \text{[Def. of } J\text{]} \end{aligned}$$

$$
\begin{aligned}
&= g(A_g\cdot, Q_g^{-1}Q_g^{-1}A_g\cdot) && [Q_g^* = Q_g, \text{ so } (Q_g^{-1})^* = Q_g^{-1}]\\
&= -g(\cdot, A_g Q_g^{-1}Q_g^{-1}A_g\cdot) && [\text{Lemma 3 (i)}]\\
&= -g(\cdot, Q_g^{-1}Q_g^{-1}A_g A_g\cdot) && [\text{Exercise 2.2 (i)}]\\
&= g(\cdot, Q_g^{-1}Q_g^{-1}Q_g^2\cdot) && [\text{Exercise 2.1 (i)}]\\
&= g.
\end{aligned}
$$

3.3. $J_g \in \mathcal{J}(V, \omega)$.

Proof:

3.3.1. $\omega(\cdot, J_g\cdot)$ is bilinear.

3.3.2. $\omega(\cdot, J_g\cdot)$ is symmetric.

Proof: It suffices to assume that $u, v \in V$ and prove that $\omega(u, J_g v) = \omega(v, J_g u)$.

$$
\begin{aligned}
\omega(u, J_g v) &= g(Q_g J_g u, J_g v) && [\text{Def. of } A_g \text{ and of } J_g]\\
&= g(J_g u, Q_g J_g v) && [\text{Exercise 2.1 (i)}]\\
&= g(Q_g J_g v, J_g u) && [g \text{ is symmetric}]\\
&= \omega(v, J_g u) && [\text{Def. of } A_g \text{ and of } J_g].
\end{aligned}
$$

3.3.3. $\omega(\cdot, J_g\cdot)$ is positive definite.

Proof: It suffices to assume that $u \in V$ and prove that $\omega(u, J_g u) \geq 0$.

$$
\begin{aligned}
\omega(u, J_g u) &= g(Q_g J_g u, J_g u) && [\text{Def. of } A_g \text{ and of } J_g]\\
&\geq 0 && [\text{Exercise 2.1 (ii)}]
\end{aligned}
$$

4. (iii)

Proof:

ω and g are compatible

$$
\begin{aligned}
&\implies A_g = (\tilde{g})^{-1} \circ \tilde{\omega} \text{ is an almost complex structure} && [\text{def. of } \omega,\, g \text{ comp.}]\\
&\implies A_g^2 = -\operatorname{id} && [\text{def. of a.c.s.}]\\
&\implies Q_g = \operatorname{id} && [\text{uniqueness of } Q_g]\\
&\implies J_g = A_g && [\text{def. of } J_g]\\
&\implies g = \omega(\cdot, A_g\cdot) = \omega(\cdot, J_g\cdot) = g_J. && [\text{def. of } A_g,\, g_J].
\end{aligned}
$$

5. (iv)

Proof:

J is ω-compatible

$$
\begin{aligned}
&\implies g_J = \omega(\cdot, J\cdot) \text{ is a metric which is compatible with } \omega && [\text{def. of } J,\, \omega \text{ comp.}]\\
&\implies J = (\tilde{g}_J)^{-1} \circ \tilde{\omega}\\
&\implies J = J_{g_J} && [\text{(iii)} + g_J,\, \omega \text{ cpt.}]. \quad \square
\end{aligned}
$$

Let $\pi\colon E \longrightarrow M$ be a vector bundle with symplectic structure ω. We will show that $\mathcal{J}(E, \omega)$ has the same homotopy type as $\mathfrak{Met}(E)$ and that $\mathfrak{Met}(E)$ is contractible. Consider the following maps:

$$\Phi\colon \mathcal{J}(E,\omega) \longrightarrow \mathfrak{Met}(E)$$
$$J \longmapsto g_J := \omega(\cdot, J\cdot),$$

$$\Psi\colon \mathfrak{Met}(E) \longrightarrow \mathcal{J}(E,\omega)$$
$$g \longmapsto J_g.$$

(Here the definitions are fiberwise, i.e. for example $\Phi(p \mapsto J_p) = (p \mapsto g_{J_p})$).

Exercise 2.3. Show that

(i) $\Phi \circ \Psi$ is homotopic to the identity;

(ii) $\Psi \circ \Phi = \mathrm{id}$.

Solution:

1. (i)

   Proof: For each g, $\Phi \circ \Psi(g) = g_{J_g}$. The map
   $$[0,1] \times \mathfrak{Met}(E) \longrightarrow \mathfrak{Met}(E)$$
   $$(t,g) \longmapsto tg + (1-t)g_{J_g}$$
   is a homotopy.

2. (ii)

   Proof: It suffices to assume that $J \in \mathcal{J}(E,\omega)$ and prove that $\Psi \circ \Phi(J) = J$.
   $$\Psi \circ \Phi(J) = J_{g_J}$$
   $$= J \qquad \text{[Exercise 2.2 (iv)]}.$$
   □

Exercise 2.4. Show that

(i) $\mathfrak{Met}(E)$ is nonempty;

(ii) $\mathfrak{Met}(E)$ is contractible.

Solution:

1. (i)

   Proof: Let $(U_\alpha, \phi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^{2n})_\alpha$ be a collection of trivializing charts for E that cover M. Let $g_\alpha \in \mathfrak{Met}(\pi^{-1}(U_\alpha))$ be the metrics on $\pi^{-1}(U_\alpha)$ defined from the Euclidean inner product using the trivialization. Let $(\rho_\alpha)_\alpha$ be a partition of unity subordinate to $(U_\alpha)_\alpha$. Then
   $$g_0 := \sum_\alpha \rho_\alpha g_\alpha \in \mathfrak{Met}(E).$$

2. (ii)

   Proof: Let g_0 be an element of $\mathfrak{Met}(E)$. The map
   $$[0,1] \times \mathfrak{Met}(E) \longrightarrow \mathfrak{Met}(E)$$
   $$(t,g) \longmapsto tg + (1-t)g_0$$
   is a homotopy. □

3 Exercise sheet No. 3 - 09-05-2019

In these exercises we are going to prove the following important theorem from symplectic geometry:

Theorem (Darboux)**.** *Let (M, ω) be a symplectic manifold and $p \in M$. Then, there exists a coordinate neighborhood $(U, q_1, \dots, q_n, p_1, \dots, p_n)$ of p such that*

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Proof sketch: Passing to coordinate neighborhoods, it suffices to assume that $\omega_0 \coloneqq \omega_{\mathrm{std}}$ and ω_1 are forms in $\mathbb{R}^{2n}$ and to prove that there exists a diffeomorphism ϕ such that $\phi^*\omega_1 = \omega_0$. We interpolate between ω_1 and ω_0,

$$\omega_t \coloneqq t\omega_1 + (1-t)\omega_0, \quad t \in [0,1],$$

and try to find a family ϕ_t such that $\phi_t^*\omega_t = \omega_0$. Denote by X_t the (time dependent) flow of ϕ_t. Such a family will necessarily satisfy

$$\begin{aligned} 0 &= \frac{d}{dt}\phi_t^*\omega_t \\ &= \phi_t^* L_{X_t}\omega_t + \phi_t^*\dot{\omega}_t \\ \implies 0 &= d\iota_{X_t}\omega_t + \dot{\omega}_t. \end{aligned}$$

Since the statement is something local we may make ω_1 exact: $\omega_1 = d\lambda_1$. Let $\lambda_t = t\lambda_1 + (1-t)\lambda_0$. Then,

$$d(\iota_{X_t}\omega_t + \dot{\lambda}_t) = 0.$$

This computation is known as the **Moser trick**, and explains why we will try to find ϕ_t as being the flow of the vector field X_t defined by

$$\iota_{X_t}\omega_t + \dot{\lambda}_t = 0.$$

□

Exercise 3.1. Let ω_0 denote the standard symplectic structure of $\mathbb{R}^{2n}$. Show that to prove the theorem it suffices to assume that $V \subset \mathbb{R}^{2n}$ is a contractible neighborhood of 0, $\omega_1 \in \Omega^2(V)$ is symplectic and $\omega_1|_0 = \omega_0|_0$, and prove that there exist $U_0, U_1 \subset V$ open and $\phi\colon U_0 \longrightarrow U_1$ a diffeomorphism such that $\phi^*\omega_1 = \omega_0$.

Solution: Let U be a coordinate neighborhood of p and $\psi\colon U \longrightarrow V'$ its corresponding diffeomorphism. Composing ψ with a translation we may assume that $0 \in V'$. There exists a basis of $\mathbb{R}^{2n}$ such that $(\psi^{-1})^*\omega|_0$ is the canonical symplectic form when written with respect to this basis. So we conclude that there exists a linear map

$$T\colon \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$

such that $((T\circ\psi)^{-1})^*\omega|_0 = \omega_0|_0$. Shrinking $V = T(V')$ we may assume that it is contractible. By hypothesis there exist $U_0, U_1 \subset V$ open and $\phi\colon U_0 \longrightarrow U_1$ a diffeomorphism such that $\phi^*((T\circ\psi)^{-1})^*\omega = \omega_0$. Then

$$\phi^{-1}\circ T\circ\psi\colon (\phi^{-1}\circ T\circ\psi)^{-1}(U_0) \longrightarrow U_0$$

is the desired coordinate neighborhood in Darboux's theorem. □

Assume then the hypothesis of Exercise 3.1.

Exercise 3.2. Define $\omega_t \coloneqq t\omega_1 + (1-t)\omega_0 \in \Omega^2(V)$ for $t \in [0,1]$. Show that

(i) $\omega_t|_0 = \omega_0|_0$;

(ii) Possibly after shrinking V, ω_t is symplectic;

(iii) There exists $\lambda_1 \in \Omega^1(V)$ such that $d\lambda_1 = \omega_1$ and $\lambda_1|_0 = \lambda_0|_0$, where λ_0 denotes the canonical symplectic potential;

(iv) $d\lambda_t = \omega_t$, where $\lambda_t \coloneqq t\lambda_1 + (1-t)\lambda_0$.

Solution:

1. (i)

   Proof:
$$\begin{aligned}\omega_t|_0 &= t\omega_1|_0 + (1-t)\omega_0|_0\\ &= t\omega_0|_0 + (1-t)\omega_0|_0\\ &= \omega_0|_0.\end{aligned}$$

2. (ii)

   Proof:

   2.1. ω_t is closed.

   Proof:
$$\begin{aligned}d\omega_t &= td\omega_1 + (1-t)d\omega_0\\ &= 0.\end{aligned}$$

   2.2. Possibly after shrinking V, ω_t is nondegenerate.

   Proof: Represent ω_t by a matrix and consider its determinant $\det(\omega_t)\colon V \longrightarrow \mathbb{R}$. Since $\omega_t|_0 = \omega_0|_0$, $\det(\omega_t)(0) \neq 0$. Since this is a continuous function, we can shrink V such that $\det(\omega_t) \neq 0$ on V. Therefore, ω_t is nondegenerate.

3. (iii)

   Proof: Since V is contractible, there exists $\lambda_1' \in \Omega^1(V)$ such that $d\lambda_1' = \omega_1$. $\lambda_1 = \lambda_1' + (\lambda_0|_0 - \lambda_1'|_0)$ is the desired form.

4. (iv)

   Proof:
$$\begin{aligned}d\lambda_t &= td\lambda_1 + (1-t)d\lambda_0\\ &= t\omega_1 + (1-t)\omega_0\\ &= \omega_t.\end{aligned}$$
□

Exercise 3.3. For each $t \in [0,1]$, let $X_t \in \mathfrak{X}(V)$ be such that $\omega_t(X_t, \cdot) = -\dot{\lambda}_t$. Show that

(i) X_t is smooth in t;

(ii) There exists $U_0 \subset V$ open and a 1-parameter family $\phi_t\colon U_0 \longrightarrow V$ of diffeomorphisms in the image such (for $t \in [0,1]$) that for all $p \in U_0$, $\phi_0(p) = p$ and $\frac{d}{dt}\phi_t(p) = X_t(\phi_t(p))$.

Solution:

1. (i)

   Proof: Because ω_t and λ_t are smooth in t.

2. (ii)

Proof: Equations $\phi_0 = \mathrm{id}$ and $\frac{d}{dt}\phi_t = X_t \circ \phi_t$ uniquely determine ϕ_t as being the (time dependent) flow of X_t. It remains to be shown that for some $U_0 \subset V$, the flow exists for $t \in [0,1]$. Since $\lambda_t|_0 = 0$, then $\dot{\lambda}_t|_0 = 0$ and $X_t|_0 = 0$. So for some U_0 small enough neighborhood of 0 the statement is true. □

Exercise 3.4. Show that U_0, $U_1 \coloneqq \phi_1(U_0)$ and $\phi \coloneqq \phi_1$ are as desired.

Solution: We need to show that $\phi_1^*\omega_1 = \omega_0$. It suffices to show that $\phi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$. For this, we show that $\frac{d}{dt}\phi_t^*\omega_t = 0$.

$$\begin{aligned}
\frac{d}{dt}\phi_t^*\omega_t &= \phi_t^* L_{X_t}\omega_t + \phi_t^*\frac{d}{dt}\omega_t && \text{[Leibniz's rule]}\\
&= \phi_t^*\left(d\iota_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) && \text{[Cartan's formula]}\\
&= \phi_t^*\left(-d\dot{\lambda}_t + \frac{d}{dt}d\lambda_t\right) && \text{[Def. of } X_t \text{ and } \lambda_t]\\
&= 0.
\end{aligned}$$

□

Exercise 3.5. There exists an analog of Darboux's theorem for contact manifolds. Write the statement of that theorem and give a proof sketch of it.

Solution:

Theorem (Contact Darboux)**.** *Let M be a $2n+1$-dimensional manifold, α be a contact form on M, and $p \in M$. There exists $(U, x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ a coordinate neighborhood of p such that*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j dy_j.$$

Proof sketch: Do the same steps as for the proof of the symplectic Darboux's theorem, but now the Moser trick computation changes slightly. Define $\alpha_t \coloneqq t\alpha_1 + (1-t)\alpha_0$. α_t is contact. We want to find ϕ_t such that $\phi_t^*\alpha_t = \alpha_0$. Let X_t be the flow of ϕ_t.

$$\begin{aligned}
0 &= \frac{d}{dt}\phi_t^*\alpha_t\\
&= \phi_t^* L_{X_t}\alpha_t + \phi_t^*\dot{\alpha}_t\\
\implies 0 &= L_{X_t}\alpha_t + \dot{\alpha}_t\\
&= \iota_{X_t}d\alpha_t + d\iota_{X_t}\alpha_t + \dot{\alpha}_t.
\end{aligned}$$

Write X_t as $X_t = H_t R_t + Y_t$, where H_t is some function, R_t is the Reeb vector field of α_t, and $Y_t \in \ker \alpha_t$. Then,

$$\begin{aligned}
0 &= \left(\iota_{X_t}d\alpha_t + d\iota_{X_t}\alpha_t + \dot{\alpha}_t\right) R_t\\
&= dH_t(R_t) + \dot{\alpha}_t(R_t)
\end{aligned}$$

determines H_t and

$$\begin{aligned}
0 &= \iota_{X_t}d\alpha_t + d\iota_{X_t}\alpha_t + \dot{\alpha}_t\\
&= \iota_{Y_t}d\alpha_t + dH_t + \dot{\alpha}_t
\end{aligned}$$

determines Y_t. (Contact Darboux)□

(Exercise 3.5)□

4 Exercise sheet No. 4 - 16-05-2019

Let m denote the Lebesgue measure of Euclidean space. In these exercises we are going to prove the following theorem.

Theorem (Sard). *Let $U \in \mathbb{R}^n$ be open, $f\colon U \longrightarrow \mathbb{R}^p$ be a smooth map, and*

$$C \coloneqq \{x \in U \mid \operatorname{rank} df(x) < p\}$$

be the set of critical points of f. Then $m(f(C)) = 0$.

We present the proof given in [MW97]. We will prove the theorem by induction on n. Note that the theorem is true for $n = 0$. Assume then that the theorem is true for $n-1$. We must now show that it is true for n. Define

$$C_i \coloneqq \left\{x \in U \mid \forall I \text{ s.t. } 0 < |I| \leq i\colon D^I f(x) = 0\right\}.$$

In other words, C_i is the set of x such that all partial derivatives of f of order $\leq i$ are 0 at x. Notice that

$$C \supset C_1 \supset C_2 \supset C_3 \supset \cdots.$$

Exercise 4.1. Show that $m(f(C - C_1)) = 0$. *Hint: since if $p = 1$ then $C = C_1$, we may assume that $p \geq 2$. Show that it suffices to assume that $x \in C - C_1$ and prove that there exists $V \subset \mathbb{R}^n$ a neighborhood of x with $m(f(V \cap C)) = 0$. Assume then that $x \in C - C_1$. Construct an appropriate diffeomorphism $h\colon V \longrightarrow V' \subset \mathbb{R}^n$ (where $V \ni x$ is the desired neighborhood) that makes the following work. $g = f \circ h^{-1}$ should map certain hyperplanes to hyperplanes. Consider the restriction of g to one of these hyperplanes and apply the induction hypothesis to the restriction. Use Fubini's theorem below to conclude about the critical set of g. And also make h such that g has the same critical points as the restriction of g to the hyperplanes.*

Solution:

1. We may assume that $p \geq 2$.

   Proof: We show that if $p = 1$ then $C = C_1$.
   $$\begin{aligned} C &= \{x \in U \mid \operatorname{rank} df(x) = 0\} && [p = 1] \\ &= \{x \in U \mid df(x) = 0\} && [p = 1] \\ &= \left\{x \in U \mid \forall I \text{ s.t. } 0 < |I| \leq 1\colon D^I f(x) = 0\right\} \\ &= C_1. \end{aligned}$$

2. It suffices to assume that $x \in C - C_1$ and prove that there exists $V \subset \mathbb{R}^n$ a neighborhood of x such that $f(V \cap C)$ has measure zero.

   Proof: For each $x \in C - C_1$ let V_x be a neighborhood of x such that $f(V_x \cap C)$ has measure zero. Pick a countable basis for $\mathbb{R}^n$ by open balls and for each x let B_x be a basis element such that $x \in B_x \subset V_x$. Then the set $\{B_x\}_{x \in C - C_1}$ is countable, $\{B_x\}_{x \in C - C_1} = \{B_{x_i}\}_{i \in \mathbb{N}}$ and covers $C - C_1$.
   $$m(f(C - C_1)) \leq m\left(f\left(\bigcup_{x \in C - C_1} B_x \cap C\right)\right) \quad [\text{the } B_x \text{ cover } C - C_1]$$

$$
\begin{aligned}
&= m\left(f\left(\bigcup_{i\in\mathbb{N}} B_{x_i} \cap C\right)\right) && [\{B_x\}_{x\in C-C_1} = \{B_{x_i}\}_{i\in\mathbb{N}}] \\
&\leq \sum_{i=1}^{\infty} m(f(B_{x_i} \cap C)) && [\text{measure of the union}] \\
&\leq \sum_{i=1}^{\infty} m(f(V_{x_i} \cap C)) && [B_{x_i} \subset V_{x_i}] \\
&= 0 && [\text{hypothesis}].
\end{aligned}
$$

3. There exist $i = 1, \dots, n$ and $j = 1, \dots, p$ such that $\frac{\partial f_j}{\partial x_i}(x) \neq 0$.

   Proof: $x \notin C_1$.

4. We may assume that $i, j = 1$.

   Proof: If not, compose f with functions that switch the coordinates. The result for the new function is equivalent to the result for f.

5. Let h be the function defined by

$$
\begin{aligned}
h \colon U &\longrightarrow \mathbb{R}^n \\
y &\longmapsto (f_1(y), y_2, \dots, y_n).
\end{aligned}
$$

6. $dh(x)$ is nonsingular.

   Proof:

$$
dh(x) = \begin{bmatrix} \frac{df_1}{dx_1}(x) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
$$

   and

$$
\begin{aligned}
\det(dh(x)) &= \frac{df_1}{dx_1}(x) \\
&\neq 0.
\end{aligned}
$$

7. There exist $V \subset \mathbb{R}^n$ a neighborhood of x and $V' \subset \mathbb{R}^n$ a neighborhood of $h(x)$ such that $h \colon V \longrightarrow V'$ is a diffeomorphism.

   Proof: Step 6 and the inverse function theorem.

8. Let $g = f \circ h^{-1} \colon V' \longrightarrow \mathbb{R}^p$.

9. Let $C' := \{y \in V' \mid \operatorname{rank} dg(y) < p\}$ be the set of critical points of g.

10. $C' = h(V \cap C)$.

    Proof:

$$
\begin{aligned}
&y \in V' \text{ and } \operatorname{rank} dg(y) < p \\
&\quad\iff h^{-1}(y) \in V \text{ and } \operatorname{rank} df(h^{-1}(y))dh^{-1}(y) < p && [\text{chain rule}] \\
&\quad\iff h^{-1}(y) \in V \text{ and } \operatorname{rank} df(h^{-1}(y)) < p && [dh^{-1}(y) \text{ is an iso. by 7}] \\
&\quad\iff h^{-1}(y) \in V \text{ and } h^{-1}(y) \in C && [\text{def. of } C] \\
&\quad\iff y \in h(V \cap C).
\end{aligned}
$$

11. $g(C') = f(V \cap C)$.

Proof:

$$\begin{aligned} g(C') &= f \circ h^{-1}(C') && \text{[def. of } g \text{ in step 8]} \\ &= f \circ h^{-1} \circ h(V \cap C) && \text{[step 10]} \\ &= f(V \cap C). \end{aligned}$$

12. For all $(t, x_2, \ldots, x_n) \in V'$, $g(t, x_2, \ldots, x_n) \in \{t\} \times \mathbb{R}^{p-1} \subset \mathbb{R}^p$.

Proof:

12.1. Let $(y_1, \ldots, y_n) = h^{-1}(t, x_2, \ldots, x_n)$.

12.2. $f_1(y) = t$.

Proof:

$$\begin{aligned} (f_1(y), y_2, \ldots, y_n) &= h(y_1, \ldots, y_n) && \text{[def. of } h\text{]} \\ &= (t, x_2, \ldots, x_n) && \text{[def. of } y \text{ in step 12.1].} \end{aligned}$$

12.3. Q.E.D.

Proof:

$$\begin{aligned} g(t, x_2, \ldots, x_n) &= f \circ h^{-1}(t, x_2, \ldots, x_n) && \text{[def. of } g \text{ in step 8]} \\ &= f(y_1, \ldots, y_n) && \text{[def. of } y \text{ in step 12.1]} \\ &= (f_1(y), f_2(y), \ldots, f_p(y)) \\ &= (t, f_2(y), \ldots, f_p(y)) && \text{[step 12.2]} \\ &\in \{t\} \times \mathbb{R}^{p-1}. \end{aligned}$$

13. The map $g^t \colon \{y \in V' \mid y_i = t\} \longrightarrow \{z \in \mathbb{R}^p \mid z_j = t\}$ defined by restricting g is well defined.

Proof: By step 12.

14. For all $t \in \mathbb{R}$ $g(C' \cap \{y \in \mathbb{R}^n \mid y_1 = t\}) = g(C') \cap \{z \in \mathbb{R}^p \mid z_1 = t\}$.

Proof:

14.1. $(\subset)$:

Proof:

$$\begin{aligned} g(C' \cap \{y \in \mathbb{R}^n \mid y_1 = t\}) &\subset g(C') \cap g(\{y \in \mathbb{R}^n \mid y_1 = t\}) && \text{[set theory fact]} \\ &\subset g(C') \cap \{z \in \mathbb{R}^p \mid z_1 = t\} && \text{[step 12].} \end{aligned}$$

14.2. $(\supset)$:

Proof:

14.2.1. It suffices to assume that $(z_1, \ldots, z_p) \in g(C') \cap \{z \in \mathbb{R}^p \mid z_1 = t\}$ and prove that there exists $(y_1, \ldots, y_n) \in C' \cap \{y \in \mathbb{R}^n \mid y_1 = t\}$ such that $g(y_1, \ldots, y_n) = (z_1, \ldots, z_p)$.

14.2.2. There exists $(y_1, \ldots, y_n) \in C'$ such that $g(y_1, \ldots, y_n) = (z_1, \ldots, z_p)$.

Proof: $(z_1, \ldots, z_p) \in g(C')$.

14.2.3. $y_1 = t$.

Proof:

$$(t, z_2, \ldots, z_p) = (z_1, z_2, \ldots, z_p) \quad [z \in \{z \in \mathbb{R}^p \mid z_1 = t\}]$$

$$= g(y_1, \dots, y_n) \qquad \text{[step 14.2.2]}$$
$$\in \{y_1\} \times \mathbb{R}^{p-1} \qquad \text{[step 12]}.$$

15. The set of critical points of g^t is $C' \cap \{y \in \mathbb{R}^n \mid y_1 = t\}$.

    Proof: Since
$$dg = \begin{bmatrix} 1 & 0 \\ * & dg^t \end{bmatrix},$$
    $y \in V'$ such that $y_i = t$ is critical for g^t if and only if it is critical for g.

16. Q.E.D.

    Proof:

    Induction hypothesis
$$\begin{aligned}
&\Longrightarrow \forall t \in \mathbb{R}\colon m_{p-1}(g^t(C' \cap \{y \in \mathbb{R}^n \mid y_1 = t\})) = 0 && \text{[step 15]}\\
&\Longrightarrow \forall t \in \mathbb{R}\colon m_{p-1}(g(C' \cap \{y \in \mathbb{R}^n \mid y_1 = t\})) = 0 && [g^t \text{ is the restriction of } g]\\
&\Longrightarrow \forall t \in \mathbb{R}\colon m_{p-1}(g(C') \cap \{z \in \mathbb{R}^p \mid z_1 = t\}) = 0 && \text{[step 14]}\\
&\Longrightarrow m_p(g(C')) = 0 && \text{[Fubini's theorem]}\\
&\Longrightarrow m_p(f(V \cap C)) = 0 && \text{[step 11]}.
\end{aligned}$$
□

Exercise 4.2. Show that $m(f(C_i - C_{i+1})) = 0$ for all $i \geq 1$. *Hint: use a similar strategy as before, choosing some local diffeomorphism h, restricting $g = f \circ h^{-1}$ to appropriate hyperplanes, and using Fubini's theorem below.*

Solution:

1. It suffices to assume that $x \in C_k - C_{k+1}$ and prove that there exists $V \subset \mathbb{R}^n$ a neighborhood of x such that $f(V \cap C_k)$ has measure zero.
2. There exist $r = 1, \dots, p$ and $s_1, \dots, s_{k+1} \in \{1, \dots, n\}$ such that $\frac{\partial^{k+1} f_r}{\partial x_{s_1} \cdots \partial x_{s_{k+1}}} \neq 0$.

   Proof: $x \in C_k - C_{k+1}$.
3. Define $w = \frac{\partial^k f_r}{\partial x_{s_2} \cdots \partial x_{s_{k+1}}}$. Then, $w(x) = 0$ and $\frac{\partial w}{\partial x_{s_1}}(x) \neq 0$.

   Proof: $x \in C_k - C_{k+1}$.
4. We may assume that $s_1 = 1$.

   Proof: If not then compose f with a map that switches coordinates. The result for this map implies the result for f.
5. Define a function h by
$$\begin{aligned} h\colon U &\longrightarrow \mathbb{R}^n \\ y &\longmapsto (w(y), y_2, \dots, y_n). \end{aligned}$$
6. There exists V a neighborhood of x and V' a neighborhood of $h(x)$ such that $h\colon V \longrightarrow V'$ is a diffeomorphism.

   Proof:
$$dh(x) = \begin{bmatrix} \frac{\partial w}{\partial x_1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$
   and the inverse function theorem.

7. $h(C_k \cap V) \subset \{0\} \times \mathbb{R}^{n-1}$.

   Proof: We must assume that $y \in C_k \cap V$ and prove that $w(y) = 0$.

   $$\begin{aligned} w(y) &= \frac{\partial^k f}{\partial x_{s_2} \cdots \partial x_{s_{k+1}}}(y) \\ &= 0. \end{aligned}$$

8. Define $g = f \circ h^{-1} \colon V' \longrightarrow \mathbb{R}^p$. Define $\bar{g} \colon (\{0\} \times \mathbb{R}^{n-1}) \cap V' \longrightarrow \mathbb{R}^p$ to be the restriction of g.

9. Every point of $h(C_k \cap V)$ is a critical point of $\bar{g}$.

   Proof:

   9.1. It suffices to assume that $z \in h(C_k \cap V)$ and prove that $d\bar{g}(z) = 0$.

   9.2. There exists $x \in C_k$ such that $h(x) = z$.

   9.3. Q.E.D.

   Proof:

   $$\begin{aligned} d\bar{g}(z) &= d\bar{g}(h(z)) && [z = h(x)] \\ &= df(x)dh^{-1}(h(x)) && [\text{chain rule}] \\ &= 0 && [\text{since } x \in C^k \text{ then } df(x) = 0]. \end{aligned}$$

10. $m(f(C_k \cap V)) = 0$.

    Proof:

    $$\begin{aligned} m(f(C_k \cap V)) &= m(\bar{g} \circ h(C_k \cap V)) && [\text{Step 7}] \\ &\leq m(\bar{g}(\text{CritPts}(\bar{g}))) && [\text{Step 9}] \\ &= m(\text{CritVal}(\bar{g})) \\ &= 0 && [\text{By the induction hypothesis}]. \end{aligned}$$ □

Exercise 4.3. Show that for k sufficiently large $m(f(C_k)) = 0$. *Hint: show that it suffices to assume that I^n is a cube with edge δ and prove that $m(f(C_k \cap I^n)) = 0$. For $x \in C_k \cap I^n$ approximate f by a Taylor polynomial. For each $r \in \mathbb{N}$, consider a subdivision of I^n into r^n cubes of edge δ/r. Use the Taylor approximation to obtain an estimate on the measure of $f(C_k \cap I^n)$ depending on r.*

Solution:

1. Let $k > n/p - 1$. It suffices to show that $m(f(C_k)) = 0$.

2. It suffices to assume that $I^n \subset U$ is a cube with edge δ, and prove that $m(f(C_k \cap I^n)) = 0$.

   Proof: C_k is covered by countably many such cubes.

3. There exists a function R and a constant c such that for all $x \in C_k \cap I^n$ and $h \in I^n - x$

   $$f(x + h) = f(x) + R(x, h)$$

   and

   $$\|R(x, h)\| \leq c\|h\|^{k+1}.$$

   Proof: Taylor's theorem and $x \in C^k$.

4. For all $r \in \mathbb{N}$, let $I_1, \ldots, I_{r^n}$ be the cubes that form a subdivision of I^n into smaller cubes of edge δ/r.

5. For all r and for each $x \in C_k \cap I^n$ let I_x be a cube of $I_1, \dots, I_{r^n}$ that contains x.
6. For all r and for all $x \in C_k \cap I^n$, $f(I_x)$ is contained in a cube of edge a/r^{k+1} where $a = 2c(\sqrt{n}\delta)^{k+1}$.

   Proof:

   6.1. It suffices to show that $f(B_{\sqrt{n}\delta/r}(x)) \subset B_R(f(x))$, where $2R = a/r^{k+1}$.

   Proof:

   $$\begin{aligned} f(I_x) &\subset f(B_{\sqrt{n}\delta/r}(x)) \\ &\subset B_R(f(x)) && \text{[hypothesis]} \\ &\subset \text{some cube of edge } 2R. \end{aligned}$$

   6.2. To show that $f(B_{\sqrt{n}\delta/r}(x)) \subset B_R(f(x))$ it suffices to assume that h is such that $\|h\| \le \sqrt{n}\delta/r$ and to prove that $\|f(x+h) - f(x)\| \le R$.

   6.3. Q.E.D.

   Proof:

   $$\begin{aligned} \|f(x+h) - f(x)\| &= \|R(x,h)\| && \text{[step 3]} \\ &\le c\|h\|^{k+1} && \text{[step 3]} \\ &\le c\left(\frac{\sqrt{n}\delta}{r}\right)^{k+1} && \text{[assumption]} \\ &= R. \end{aligned}$$

7. For all r, $m(f(C_k \cap I^n)) \le a^p r^{n-(k+1)p}$.

   Proof:

   $$\begin{aligned} m(f(C_k \cap I^n)) &\le r^n \sup_{x \in C_k} m(f(I_x)) && \left[f(C_k \cap I^n) \subset f\left(\textstyle\bigcup_{x \in C_k} I_n\right)\right] \\ &\le r^n \left(\frac{a}{r^{k+1}}\right)^p && \text{[step 6]} \\ &= a^p r^{n-p(k+1)}. \end{aligned}$$

8. Q.E.D.

   Proof: For each $r \in \mathbb{N}$, $m(f(C_k \cap I^n)) \le a^p r^{n-(k+1)p}$, therefore $m(f(C_k \cap I^n)) = 0$. □

Exercise 4.4. Conclude the proof of Sard's theorem.

Solution: Let k be as in Exercise 4.3.

$$\begin{aligned} &m(f(C)) \\ &= m\left(f\left((C - C_1) \cup \bigcup_{i=1}^{k} (C_i - C_{i+1}) \cup C_{k+1}\right)\right) && \text{[write } C \text{ as a union of its subsets]} \\ &\le m(f(C - C_1)) \\ &\quad + \sum_{i=1}^{k} m(f(C_i - C_{i+1})) \\ &\quad + m(f(C_{k+1})) && \text{[additivity of measures]} \\ &= 0 + 0 + 0 && \text{[Exercises 4.1, 4.2 and 4.3]} \\ &= 0. \end{aligned}$$

□

Theorem (Fubini). *Let $A \subset \mathbb{R}^p = \mathbb{R}^1 \times \mathbb{R}^{p-1}$ be a measurable set. If for all $x \in \mathbb{R}$ we have that $m_{p-1}(A \cap x \times \mathbb{R}^{p-1}) = 0$ then $m_p(A) = 0$.*

5 Exercise sheet No. 5 - 23-05-2019

Let M be a compact $(2n+1)$-dimensional manifold without boundary and λ be a contact form on M. Let X be the Reeb vector field of (M, λ). Let $W = \mathbb{R} \times M$ be the symplectization of M, and $\tilde{J}$ be an SFT-like almost complex structure on W. In exercise sheets 5 and 6 we are going to prove the following theorem by Hofer:

Theorem (Hofer)**.** *Let $\tilde{u}\colon \mathbb{C} \longrightarrow \mathbb{R}\times M$ be such that $\tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0$ and $0 < E(\tilde{u}) < \infty$. Let $T := \int_{\mathbb{C}} u^* d\lambda$. Then $T > 0$ and for every sequence $0 < R'_k \longrightarrow \infty$ there exists a subsequence $(R_k)_{k\in\mathbb{N}}$ and x a T-periodic solution of $\dot{x}(t) = X(x(t))$ such that $u\left(R_k e^{\frac{2\pi i}{T}t}\right)$ converges (as a function of t) in the C^∞-topology to $x(t)$.*

The proof that we give is the one given in [AH19]. For hints on how to solve each exercise, read the proof sketches in the solutions.

Exercise 5.1 (Hofer lemma)**.** Let (X, d) be a complete metric space. Show that for every $f\colon X \longrightarrow [0, +\infty)$ continuous, $\varepsilon_0 > 0$, $x_0 \in X$, there exist $\varepsilon \in (0, \varepsilon_0]$ and $x \in X$ such that

(i) $\varepsilon f(x) \geq \varepsilon_0 f(x_0)$,

(ii) $d(x, x_0) \leq 2\varepsilon$,

(iii) $\forall y \in \overline{B_\varepsilon(x)}\colon f(y) \leq 2f(x)$.

Proof sketch: If (iii) holds for $\varepsilon = \varepsilon_0$, $x = x_0$ then we are done. Otherwise let $\varepsilon_1 = \varepsilon_0/2$ and choose x_1 such that $d(x_1, x_0) \leq \varepsilon_0$ and $f(x_1) > 2f(x_0)$. If (iii) holds for $\varepsilon = \varepsilon_1$, $x = x_1$ then we are done. Otherwise continue with this procedure. Use the completeness hypothesis to show that this procedure must stop.
Solution:
1. Consider the following algorithm:

For $k = 0$,	if	ε_0, x_0 satisfy (iii), then stop
	otherwise	let $\varepsilon_1 = \varepsilon_0/2$ and
		choose x_1 such that $d(x_1, x_0) \leq \varepsilon_0$ and $f(x_1) > 2f(x_0)$
For $k = 1$,	if	ε_1, x_1 satisfy (iii), then stop
	otherwise	let $\varepsilon_2 = \varepsilon_1/2$ and
		choose x_2 such that $d(x_2, x_1) \leq \varepsilon_1$ and $f(x_2) > 2f(x_1)$
...	...	...

2. The algorithm stops.

   Proof:

   2.1. Assume by contradiction that the algorithm does not stop

   2.2. There exists a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ such that $d(x_k, x_{k-1}) \leq \varepsilon_{k-1}$ and $f(x_k) > 2f(x_{x_k-1})$.

   Proof: By assumption in step 2.1.

   2.3. The sequence $(x_k)_{k\in\mathbb{N}}$ is Cauchy.

Proof:

$$
\begin{aligned}
d(x_{k+l}, x_k) &\leq \sum_{j=k}^{k+l-1} d(x_{j+1}, x_j) && \text{[triangular ineq.]} \\
&\leq \sum_{j=k}^{\infty} \varepsilon_j && \text{[step 2.1]} \\
&= \sum_{j=k}^{\infty} \frac{\varepsilon_0}{2^j} && \text{[step 1]} \\
&= \frac{\varepsilon_0}{2^{k-1}} && \text{[lim. of geometric series].}
\end{aligned}
$$

2.4. The sequence $(x_k)_{k\in\mathbb{N}}$ has a limit $x_\infty \in X$.

Proof: Step 2.3 and X is complete.

2.5. $\lim_{k\to\infty} f(x_k) = f(x_\infty) \in \mathbb{R}^+$.

Proof: f is continuous.

2.6. $\lim_{k\to\infty} f(x_k) = \infty$.

Proof: Since $f(x_k) > 2f(x_{k-1})$ then $f(x_{k+1}) > 2^k f(x_1)$ and by $f(x_1) > 2f(x_0) \geq 0$ the result follows.

2.7. Q.E.D.

Proof: Steps 2.5 and 2.6 give a contradiction.

3. Let k be the iteration where the algorithm stops. Let $\varepsilon = \varepsilon_k$ and $x = x_k$.
4. (i)

Proof:

$$
\begin{aligned}
\varepsilon f(x) &= \varepsilon_k f(x_k) \\
&> \varepsilon_k 2^k f(x_0) \\
&= \varepsilon_0 f(x_0).
\end{aligned}
$$

5. (ii)

Proof:

$$
\begin{aligned}
d(x, x_0) &= d(x_k, x_0) \\
&\leq \sum_{j=0}^{k-1} \varepsilon_j \\
&\leq \sum_{j=0}^{\infty} \frac{\varepsilon_0}{2^j} \\
&= 2\varepsilon_0.
\end{aligned}
$$

6. (iii)

Proof: $x = x_0$, $\varepsilon = \varepsilon_k$ satisfy (iii) because the algorithm stops at k.

7. Q.E.D.

Proof: Steps 4, 5 and 6 show that ε and x defined in step 3 are as desired. □

Exercise 5.2. Let $\tilde{u} = (a, u) \colon \mathbb{C} \longrightarrow \mathbb{R} \times M$ be a $\tilde{J}$-holomorphic map such that $E(\tilde{u}) < +\infty$ and $\int_{\mathbb{C}} u^* d\lambda = 0$. Show that $\tilde{u}$ is constant.

Proof sketch: Use $\int_{\mathbb{C}} u^* d\lambda = 0$ to show that there exists an f such that $df = u^*\lambda$. Then $\Phi := a + if \colon \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic. So we have replaced u by a holomorphic map $\Phi \colon \mathbb{C} \longrightarrow \mathbb{C}$ such that $E(\Phi) = E(u)$. We now show that Φ has uniformly bounded 1st derivatives. Assume otherwise, by contradiction. There exists a sequence $(z_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ such that $R_k := |\nabla \Phi(z_k)| \longrightarrow +\infty$. Define $\Phi_k(z) = \Phi\left(z_k + \frac{z}{R_k}\right) - \Phi(z_k)$. So as k increases, Φ_k is given by evaluation Φ on smaller and smaller regions near z_k. Using the Hofer lemma, upgrade the sequences z_k, R_k such that now an argument using the Arzelà-Ascoli theorem shows that Φ_k converges to some $\Psi \colon \mathbb{C} \longrightarrow \mathbb{C}$ in the C^∞_W-topology. Show that Ψ must be a biholomorphism using the properties of the limit. So it has infinite energy. Since Ψ is obtained from Φ by looking at smaller and smaller open sets in the domain of Φ and rescaling them biholomorphically, $E(\Psi) \leq E(\Phi) < \infty$. Contradiction. So Φ has bounded derivatives. By Liouville Φ is affine. By finite energy Φ cant be a biholomorphism, hence it is constant. So u is constant.

Solution:

1. There exists an $f \in C^\infty(\mathbb{C}, \mathbb{R})$ such that $df = u^*\lambda$.

   Proof:

   1.1. $u^* d\lambda = \frac{1}{2}\left(|\pi_\lambda u_s|^2_J + |\pi_\lambda u_t|^2_J\right) ds \wedge dt$.

   Proof: Compute the form in local coordinates.

   1.2. $d(u^*\lambda) = 0$.

   Proof:

$$\begin{aligned} 0 &= \int_{\mathbb{C}} u^* d\lambda && \text{[hypothesis]} \\ &= \frac{1}{2} \int_{\mathbb{C}} \frac{1}{2} \left(|\pi_\lambda u_s|^2_J + |\pi_\lambda u_t|^2_J\right) ds \wedge dt && \text{[step 1.1]}, \end{aligned}$$

   therefore, $|\pi_\lambda u_s|^2_J = 0$ and $|\pi_\lambda u_t|^2_J = 0$. By step 1.1, $d(u^*\lambda) = u^* d\lambda = 0$.

   1.3. Q.E.D.

   Proof: $u^*\lambda$ is a closed form in $\mathbb{C}$ by step 1.2, hence it is exact.

2. Define $\Phi = a + if \colon \mathbb{C} \longrightarrow \mathbb{C}$. Then Φ is holomorphic.

   Proof:

   2.1. $df = -da \circ i$.

   Proof:

$$\begin{aligned} df &= u^*\lambda && \text{[step 1]} \\ &= -da \circ i && \text{[projection of the } \tilde{J}\text{-holomorphic} \\ & && \quad \text{curve eq. for } \tilde{u} \text{ onto } \mathbb{R} \oplus \langle X \rangle]. \end{aligned}$$

   2.2. Φ satisfies the Cauchy-Riemann equations.

   Proof:

$$\begin{aligned} i \circ d\Phi &= i \circ da - df \\ &= i \circ ds + da \circ i && \text{[step 2.1]} \\ &= da \circ i + i \circ df \circ i && \text{[step 2.1]} \\ &= d\Phi \circ i. \end{aligned}$$

2.3. Q.E.D.

Proof: Step 2.2 and Φ is C^∞.

3. For each $\phi \in \Sigma$ define $\tau_\phi = d(\phi dt) \in \Omega^2(\mathbb{C})$. Then, for all $\phi \in \Sigma$,
$$\int_{\mathbb{C}} \Phi^* \tau_\phi = \int_{\mathbb{C}} \tilde{u}^* d(\phi\lambda) \le E_\Sigma(\tilde{u}).$$
Proof:
$$\begin{aligned} E_\Sigma(\tilde{u}) &\ge \int_{\mathbb{C}} \tilde{u}^* d(\phi\lambda) && \text{[def. of Energy]} \\ &= \int_{\mathbb{C}} d(\phi(a) u^* \lambda) \\ &= \int_{\mathbb{C}} d(\phi(a) df) && \text{[step 1]} \\ &= \int_{\mathbb{C}} \Phi^* d(\phi(s) dt) \\ &= \int_{\mathbb{C}} \Phi^* \tau_\phi && \text{[def. of } \tau_\phi\text{]}. \end{aligned}$$

4. Φ has uniformly bounded first derivatives.

Proof sketch: Argue by contradiction. Show that if the derivatives of Φ explode then Φ cant have finite Energy, using a "Bubbling-off" argument.
Proof:

4.1. Assume by contradiction that Φ does not have uniformly bounded first derivatives.

4.2. There exists a sequence $(z_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $R_k := |\nabla\Phi(z_k)| \longrightarrow +\infty$.

Proof: Step 4.1.

4.3. Let $\varepsilon_k := \frac{\ln(R_k)}{R_k}$. Then $0 < \varepsilon_k \longrightarrow 0$ and $\varepsilon_k R_k \longrightarrow +\infty$.

4.4. There exist sequences $\varepsilon'_k \in (0, \varepsilon_k]$, $z'_k \in \mathbb{C}$ such that

(i) $\varepsilon'_k |\nabla\Phi(z'_k)| \ge \varepsilon_k |\nabla\Phi(z_k)|$

(ii) $|z_k - z'_k| \le 2\varepsilon'_k$

(iii) $\forall z \in B_{\varepsilon'_k}(z'_k)$: $|\nabla\Phi(z)| \le |\nabla\Phi(z'_k)|$

Proof: Hofer's lemma applied to $X = \mathbb{C}$, $f = |\nabla\Phi|$, $x_0 = z_k$, $\varepsilon_0 = \varepsilon_k$, giving $x = z'_k$ and $\varepsilon = \varepsilon'_k$.

4.5. Let $R'_k := |\nabla\Phi(z'_k)|$. The sequences ε'_k, z'_k satisfy:

(i) $R'_k \varepsilon'_k \longrightarrow +\infty$

(ii) $0 < \varepsilon'_k \longrightarrow 0$

(iii) $R'_k \longrightarrow +\infty$

Proof:

4.5.1. (i)

Proof:
$$\begin{aligned} R'_k \varepsilon'_k &= \varepsilon'_k |\nabla\Phi(z'_k)| && \text{[def. of } R'_k\text{]} \\ &\ge \varepsilon_k |\nabla\Phi(z_k)| && \text{[step 4.4 (i)]} \\ &= \varepsilon_k R_k && \text{[def. of } R_k\text{]} \\ &\longrightarrow +\infty && \text{[step 4.3]} \end{aligned}$$

4.5.2. (ii)

Proof: By step 4.3, $\varepsilon'_k < \varepsilon_k \longrightarrow 0$.

4.5.3. (iii)

Proof: By steps 4.5.1 and 4.5.2.

4.6. Define

$$\begin{aligned}\Phi_k \colon \mathbb{C} &\longrightarrow \mathbb{C}\\ z &\longmapsto \Phi\left(z'_k + \frac{z}{R'_k}\right) - \Phi(z'_k).\end{aligned}$$

4.7. The maps Φ_k satisfy

(i) $|\nabla\Phi_k(0)| = 1$

(ii) $|\Phi_k(0)| = 0$

(iii) $\forall z \in B_{\varepsilon'_k R'_k(0)} \colon |\nabla\Phi_k(z)| \le 2$

Proof:

4.7.1. (i)

Proof:

$$\begin{aligned}|\nabla\Phi_k(0)| &= |\frac{1}{R'_k}\nabla\Phi(z'_k)| && \text{[def. of } \Phi_k \text{ and chain rule]}\\ &= 1 && \text{[def. of } R'_k\text{]}.\end{aligned}$$

4.7.2. (ii)

Proof:

$$\begin{aligned}\Phi_k(0) &= \Phi(z'_k) - \Phi(z'_k) && \text{[def. of } \Phi_k\text{]}\\ &= 0.\end{aligned}$$

4.7.3. (iii)

Proof: It suffices to assume that $z \in B_{\varepsilon'_k R'_k(0)}$, and prove that $|\nabla\Phi_k(z)| \le 2$.

$$\begin{aligned}z \in B_{\varepsilon'_k R'_k(0)} &\Longleftrightarrow \frac{1}{R'_k} z \in B_{R'_k}(0)\\ &\Longleftrightarrow z'_k + \frac{1}{R'_k z} \in B_{\varepsilon'_k}(z'_k)\\ &\Longrightarrow \left|\nabla\Phi\left(z'_k + \frac{1}{R'_k z}\right)\right| \le 2|\nabla\Phi(z'_k)| && \text{[step 4.4 (iii)]}\\ &\Longleftrightarrow |\nabla\Phi_k(z'_k)| \le 2 && \text{[chain rule]}.\end{aligned}$$

4.8. There exists a subsequence of Φ_k (whose index we still denote by k) and $\Psi \colon \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic such that Φ_k converges in the C^∞_W topology to Ψ.

Proof:

4.8.1. For all $K \subset \mathbb{C}$ and $l \in \mathbb{N}_0$ compact there exists a $C_{K,l} > 0$ such that for all $k \in \mathbb{N}$

$$\left\|\frac{d^l\Phi_k}{dz^l}\right\|_{\max,K} < C_{K,l}.$$

Proof: Use the gradient bound of step 4.7 (iii) and $\Phi_k(0) = 0$ (step 4.7 (ii)) to get a uniform C^0-bound. Use Cauchy's integral formula and the uniform C^0-bound to get uniform C^∞-bounds.

4.8.2. For all $K \subset \mathbb{C}$ compact there exists a subsequence $(\Phi_{k_{j,K}})_{j\in\mathbb{N}}$ of Φ_k and $\Phi_K \in C^\infty(K, \mathbb{C})$ a holomorphic map such that $\Phi_{k_{j,K}}$ converges to Φ_K in the C^∞-topology.

Proof: By step 4.8.1, for all $l \in \mathbb{N}$ the sequence $\Phi_k^{(l)}$ is uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem applied to $\Phi_k^{(0)}$, there exists $\Phi_K^0 \in C^0(K;\mathbb{C})$ and a subsequence of $\Phi_k^{(0)}$ that converges uniformly to Ψ_K^0. By the Arzelà-Ascoli theorem applied to $\Phi_k^{(1)}$, there exists $\Psi_K^1 \in C^0(K;\mathbb{C})$ and a further subsequence of $\Phi_k^{(1)}$ that converges uniformly to Ψ_K^1. Since

$$\begin{aligned}\Psi_K^1 &= \lim \Phi_k^1 \\ &= \frac{d}{dz} \lim \Phi_k^{(0)} \\ &= \frac{d}{dz}(\Psi_K^0)\end{aligned}$$

we conclude that $\Psi_K := \Psi_K^0$ is differentiable. Repeating this for $l = 2, 3, \ldots$ we conclude that $\Psi_K \in C^\infty(K, \mathbb{C})$ and we get for each l a further subsequence of the previous one. The desired final subsequence is that whose l-th term is the l-th term of the l-th subsequence. This final subsequence is such that Φ_k converges uniformly with all derivatives to Ψ_K. So Ψ_K is holomorphic.

4.8.3. Q.E.D.

Proof: For $K = \overline{B_1(0)}$, use step 4.8.2 to take a subsequence of Φ_k that converges uniformly with all derivatives to $\Psi^1 \in C^\infty(\overline{B_1(0)}, \mathbb{C})$ holomorphic. For $K = \overline{B_2(0)}$, use step 4.8.2 to take a further subsequence of Φ_k that converges uniformly with all derivatives to $\Psi^2 \in C^\infty(\overline{B_2(0)}, \mathbb{C})$ holomorphic. Then $\Psi^2|_{\overline{B_1(0)}} = \Psi^1$. Repeat this procedure for each radii in $\mathbb{N}$. We get a map $\Psi \in C^\infty(\mathbb{C}, \mathbb{C})$ which is holomorphic. Consider the subsequence whose l-th term is the l-th term of the l-th subsequence that we created. Then this final subsequence Φ_k converges uniformly with all derivatives to Ψ.

4.9. Ψ satisfies $|\nabla\Psi(0)| = 1$ and $|\nabla\Psi(z)| \leq 2$ on $\mathbb{C}$.

Proof: Steps 4.7 and 4.8.

4.10. Ψ is of the form $\Psi(z) = \alpha z + \beta$ for $\alpha \neq 0$, $\alpha, \beta \in \mathbb{C}$.

Proof: Ψ is holomorphic and $|\nabla\Psi(z)| \leq 2$ on $\mathbb{C}$ imply by Liouville's theorem that it is affine. $|\nabla\Psi(0)| = 1$ implies nonzero derivative.

4.11. Choose $\phi \in \Sigma$ nonconstant and let $\phi_k(s) := \phi(s - \mathrm{Re}(\Phi(z_k')))$.

4.12. $\int_{B_{\varepsilon_k' R_k'}(0)} \Phi_k^* \tau_\phi \leq E(\tilde{u}) < \infty$.

Proof:

$$\int_{B_{\varepsilon_k' R_k'}(0)} \Phi_k^* \tau_\phi = \int_{B_{\varepsilon_k' R_k'}(z_k')} \Phi^* \tau_{\phi_k} \quad \text{[change of variables]}$$

$$\begin{aligned} &\leq \int_{\mathbb{C}} \Phi^* \tau_{\phi_k} && \text{[positive integrand]} \\ &= \int_{\mathbb{C}} \tilde{u}^* d(\phi_k \lambda) && \text{[step 3]} \\ &\leq E(\tilde{u}) && [\phi_k \in \Sigma] \\ &< \infty && \text{[hypothesis].} \end{aligned}$$

4.13. $E(\tilde{u}) \geq \int_{\mathbb{C}} \Psi^* \tau_\phi$.

Proof: It suffices to show that for all $R > 0$, $E(\tilde{u}) \geq \int_{B_R(0)} \Psi^* \tau_\phi$.

$$\begin{aligned} E(\tilde{u}) &\geq \lim_{k \to \infty} \int_{B_{\varepsilon'_k R'_k}(0)} \Phi_k^* \tau_\phi && \text{[step 4.12]} \\ &\geq \lim_{k \to \infty} \int_{B_R(0)} \Phi_k^* \tau_\phi && [\varepsilon'_k R'_k \to \infty] \\ &= \int_{B_R(0)} \Psi^* \tau_\phi && \text{[step 4.8].} \end{aligned}$$

4.14. Q.E.D.

Proof:

$$\begin{aligned} +\infty > E(\tilde{u}) && \text{[hypothesis]} \\ \geq \int_{\mathbb{C}} \Psi^* \tau_\phi && \text{[step 4.13]} \\ = \int_{\mathbb{C}} \tau_\phi && [\Psi \text{ is a biholomorphism}] \\ = +\infty && [\phi \text{ is nonconstant}] \end{aligned}$$

gives a contradiction.

5. Φ is of the form $\Phi(z) = \alpha z + \beta$, $\alpha, \beta \in \mathbb{C}$.

   Proof: Φ is holomorphic, step 4 and Liouville's theorem.

6. $\alpha = 0$.

   Proof: Assume otherwise. Then Φ is biholomorphic. Choose $\phi \in \Sigma$ nonconstant.

$$\begin{aligned} +\infty > E(\tilde{u}) && \text{[hypothesis]} \\ \geq \int_{\mathbb{C}} \tilde{u}^* d(\phi\lambda) && [\phi \in \Sigma] \\ = \int_{\mathbb{C}} \Phi^* \tau_\phi && \text{[step 3]} \\ = \int_{\mathbb{C}} \tau_\phi && [\Phi \text{ is a biholomorphism}] \\ = +\infty && [\phi \text{ is nonconstant}] \end{aligned}$$

   gives a contradiction.

7. Q.E.D.

   Proof:

$$\begin{aligned} \text{Steps } 5, 6 &\implies \Phi \text{ is constant} \\ &\implies a, f \text{ are constant} \\ &\implies \tilde{u} \text{ is constant.} \end{aligned}$$

$\square$

6 Exercise sheet No. 6 - 06-06-2019

Let M be a compact $(2n+1)$-dimensional manifold without boundary and λ be a contact form on M. Let X be the Reeb vector field of (M,λ). Let $W = \mathbb{R}\times M$ be the symplectization of M, and $\tilde{J}$ be an SFT-like almost complex structure on W. In exercise sheets 5 and 6 we are going to prove the following theorem by Hofer:

Theorem (Hofer)**.** *Let $\tilde{u}\colon \mathbb{C}\longrightarrow \mathbb{R}\times M$ be such that $\tilde{u}_s+\tilde{J}(\tilde{u})\tilde{u}_t = 0$ and $0 < E(\tilde{u}) < \infty$. Let $T := \int_{\mathbb{C}} u^* d\lambda$. Then $T > 0$ and for every sequence $0 < R'_k \longrightarrow \infty$ there exists a subsequence $(R_k)_{k\in\mathbb{N}}$ and x a T-periodic solution of $\dot{x}(t) = X(x(t))$ such that $u\left(R_k e^{\frac{2\pi i}{T}t}\right)$ converges (as a function of t) in the C^∞-topology to $x(t)$.*

The proof that we give is the one given in [AH19]. This book contains the following analytical result that we are going to need:

Theorem (C^∞-bounds)**.** *For each $c > 0$, let*

$$\Gamma(c) = \left\{\tilde{u}\colon D \longrightarrow \mathbb{R}\times M \mid \tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0, |\tilde{u}_s|^2 + |\tilde{u}_t|^2 \le c^2\right\}.$$

Then, for each $c > 0$, $\delta \in (0,1)$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| \ge 1$ there exists $d > 0$ such that for all $\tilde{u}\in\Gamma(c)$ we have that

$$\|D^\alpha \tilde{u}\|_{C^0(\overline{B_\delta(0)})} \le d.$$

For hints on how to solve each exercise, read the proof sketches in the solutions.

Exercise 6.1. Let $\tilde{v}\colon \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times M$ be such that $\tilde{v}_s + \tilde{J}(\tilde{v})\tilde{v}_t = 0$, $E(\tilde{v}) < \infty$ and $\int_{\mathbb{C}} v^* d\lambda > 0$. Show that there exists $c > 0$ such that for all $(s,t)\in\mathbb{R}\times S^1$, $|\nabla\tilde{v}(s,t)| \le c$.

Proof sketch: Assume by contradiction that such a c does not exist. Extend $\tilde{v}$ to $\tilde{v}\colon \mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}\times M$ periodically in the second argument. There exists a sequence $z_k = (s_k,t_k)$ such that $R_k := |\nabla\tilde{v}(s_k,t_k)| \longrightarrow +\infty$. Let

$$\tilde{v}_k := \left(b\left(z_k + \frac{z}{R_k}\right) - b(z_k), v\left(z_k + \frac{z}{R_k}\right)\right).$$

So as k increases $\tilde{v}_k$ is given by evaluating v on smaller and smaller regions around z_k. Using the Hofer lemma, upgrade the sequences (s_k,t_k), R_k such that now the C^∞-bounds theorem and an argument using the Arzelà-Ascoli theorem shows that $\tilde{v}_k$ converges to some $\tilde{w} := \mathbb{C}\longrightarrow\mathbb{C}$ in the C^∞_W-topology. By the previous exercise and $\int_{\mathbb{C}} w^* d\lambda = 0$ then w is constant. By $|\nabla w(0)| = 1$ w is nonconstant. Contradiction.

Solution:

1. Assume by contradiction that such a c does not exist. Consider the function $\tilde{v}\colon \mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}\times M$ that is obtained by extending $\tilde{v}$ periodically in the second argument.
2. There exist sequences $(s_k,t_k)\in\mathbb{R}\times S^1$ and $\varepsilon_k\in\mathbb{R}^+$ such that
   (i) $R_k := |\nabla\tilde{v}(s_k,t_k)| \longrightarrow \infty$
   (ii) $\varepsilon_k \longrightarrow 0$
   (iii) $\varepsilon_k R_k \longrightarrow +\infty$
   (iv) $|\nabla\tilde{v}(s,t)| \le 2|\nabla\tilde{v}(s_k,t_k)|$ for $(s-s_k)^2 + (t-t_k)^2 \le \varepsilon_k^2$
   (v) $s_k \longrightarrow +\infty$

Proof: By assumption, there exists a sequence (s_k, t_k) satisfying (*i*). Let $\varepsilon_k = \frac{\ln(R_k)}{R_k}$. Then (i), (ii), (iii) are satisfied. Use Hofer's lemma with $X = S^1 \times \mathbb{R}$, $f = |\nabla \tilde{v}|$, $x_0 = (s_k, t_k)$, $\varepsilon_0 = \varepsilon_k$ to get new $(s_k, t_k) := x$, $\varepsilon_k = \varepsilon$. These satisfy (i), (ii), (iii), (iv). Then since (s_k, t_k) satisfies (i), it also satisfies (v).

3. Define $z_k := (s_k, t_k)$ and
$$\begin{aligned}\tilde{v}_k &:= (b_k(z), v_k(z)) \\ &:= \left(b\left(z_k + \frac{z}{R_k}\right) - b(z_k), v\left(z_k + \frac{z}{R_k}\right)\right).\end{aligned}$$

4. $\tilde{v}_k$ satisfies
   (i) $|\nabla \tilde{v}_k(0)| = 1$
   (ii) $|\nabla \tilde{v}_k(z)| \leq 2$ for $z \in B_{\varepsilon_k R_k}(0)$
   (iii) $\tilde{v}_k$ is $\tilde{J}$-holomorphic on $\mathbb{C}$
   (iv) $\forall \varphi \in \Sigma$: $\int_{B_{\varepsilon_k R_k}(0)} \tilde{v}_k^* d(\varphi\lambda) \leq E(\tilde{v}) < +\infty$
   (v) $\forall R > 0$: $\lim_{k \to +\infty} \int_{B_R(0)} v_k^* d\lambda = 0$

   Proof:

   4.1. (i)

   Proof: By definition of $\tilde{v}_k$ and the chain rule.

   4.2. (ii)

   Proof: By step 2 (iv), definition of $\tilde{v}_k$ and the chain rule.

   4.3. (iii)

   Proof: $\tilde{v}_k$ is a composition of holomorphic functions.

   4.4. (iv)

   Proof: Let $\varphi_k(s) = \varphi(s - b(z_k))$.
$$\begin{aligned}\int_{B_{\varepsilon_k R_k}(0)} \tilde{v}_k^* d(\varphi\lambda) &= \int_{B_{\varepsilon_k}(z_k)} \tilde{v} d(\varphi_k \lambda) && \text{[change of variables]} \\ &\leq \int_{\mathbb{R}\times[0,1]} \tilde{v}^* d(\varphi_k\lambda) \\ &\leq E(\tilde{v}) && [\varphi_k \in \Sigma] \\ &< +\infty && \text{[hypothesis]}.\end{aligned}$$

   4.5. (v)

   Proof: Since
$$\begin{aligned}\int_{\mathbb{R}\times[0,1]} v^* d\lambda &\leq E(\tilde{v}) && [1 \in \Sigma] \\ &< +\infty && \text{[hypothesis]},\end{aligned}$$
   then
$$\begin{aligned}\lim_{k\to\infty} \int_{B_R(0)} v_k^* d\lambda &= \lim_{k\to\infty} \int_{B_{R/R_k}(z_k)} v^* d\lambda && \text{[change of variables]} \\ &= 0.\end{aligned}$$

5. There exists a subsequence of $\tilde{v}_k$ (whose index we still denote by k) and
$$\tilde{w} = (\beta, w) \colon \mathbb{C} \longrightarrow \mathbb{R} \times M$$
such that $\tilde{v}_k$ converges in $C^\infty_W(\mathbb{C}, \mathbb{R} \times M)$ to $\tilde{w}$.

Proof: By 4, the C^∞-bounds theorem above, and an application of the Arzelà-Ascoli theorem similar to the one in the proof of exercise 5.2.

6. $\tilde{w}$ satisfies
   (i) $|\nabla \tilde{w}(0)| = 1$,
   (ii) $|\nabla \tilde{w}(z)| \leq 2$ on $\mathbb{C}$,
   (iii) $\tilde{w}$ is holomorphic on $\mathbb{C}$,
   (iv) $E(\tilde{w}) \leq E(\tilde{v}) < +\infty$,
   (v) $\int_{\mathbb{C}} w^* d\lambda = 0$.

   Proof: Step 4 (i), (ii), (iii), (iv), (v) and step 5.

7. Q.E.D.

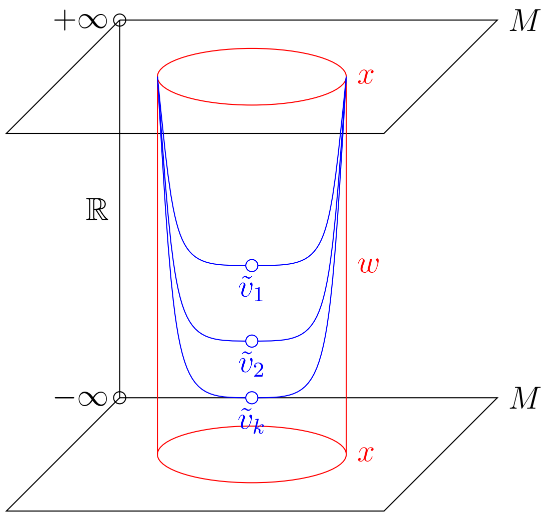
   Proof: By $\int_{\mathbb{C}} w^* d\lambda = 0$ and exercise 5.2, w is constant. By $|\nabla w(0)| = 1$, w is nonconstant. Contradiction. □

Exercise 6.2. Prove Hofer's theorem.

Proof sketch: Use the biholomorphism

$$\begin{aligned} \phi \colon \mathbb{R} \times S^1 &\longrightarrow \mathbb{C} \backslash \{0\} \\ (s,t) &\longmapsto e^{2\pi(s+it)} \end{aligned}$$

to write the result in terms of a map $\tilde{v} = \tilde{u} \circ \phi \colon \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times M$. In these terms we want to show that for each sequence $s_k \in \mathbb{R}$ going to $+\infty$ there exists a subsequence such that $\lim^{C^\infty}_{k \to \infty} v(s_k, \cdot)$ is a Reeb orbit. As before, we are going to do some analysis to get a C^∞ limit of a sequence of functions, and then we prove that this limit map defines a Reeb orbit. Exercise 6.1 gives us gradient bounds for $\tilde{v}$. Define $\tilde{v}_k(s,t) = (b(s+s_k,t) - b(s_k,0), v(s_k+s,t))$. So as k increases, $\tilde{v}_k$ is given by evaluating v on circles with higher and higher s coordinate. We point out that if the result of the theorem is true, then a suitable subsequence of $\tilde{v}_k$ should converge to a cylinder of Reeb orbits. See the figure below.



Again using the C^∞-bounds theorem and the Arzelà-Ascoli theorem we can find out that a subsequence of $\tilde{v}_k$ converges C^∞_W to some $\tilde{w} \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times M)$. Like in exercise 5.2

use $\int_{\mathbb{C}} w^* d\lambda = 0$ to find f such that $df = w^* d\lambda$ and replace w by $\Phi = \beta + if : \mathbb{C} \longrightarrow \mathbb{C}$. Use these new maps and the properties of $\tilde{v}$ and $\tilde{w}$ to show that $\tilde{w}$ must really be a cylinder of Reeb orbits.

Solution:

1. It suffices to assume that $\tilde{v} = (b, v) : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times M$ satisfies
   (i) $\tilde{v}$ is holomorphic on $\mathbb{R} \times S^1$,
   (ii) $0 < E(\tilde{v}) < +\infty$,
   (iii) $\exists p \in M : \forall t \in S^1 : \lim_{s \to -\infty} v(s, t) = p$,
   (iv) $T := \int_{\mathbb{R} \times S^1} v^* d\lambda$,
   (v) $s_k \in \mathbb{R}$ is a sequence converging to $+\infty$

   and to prove that there exists a subsequence of $(s_k)_{k \in \mathbb{N}}$ (whose index we still denote by k) and x a T-periodic solution of $x(t) = X(x(t))$ such that $v(s_k, \cdot)$ converges to $x(T \cdot)$ in the C^∞-topology.

   Proof: By exercise 5.2, $T := \int_{\mathbb{C}} u^* d\lambda > 0$. Consider the map
   $$\begin{aligned} \phi : \mathbb{R} \times S^1 &\longrightarrow \mathbb{C} \backslash \{0\} \\ (s, t) &\longmapsto e^{2\pi(s+it)}, \end{aligned}$$
   and define $\tilde{v} = \tilde{u} \circ \phi$, $e^{2\pi s_k'} := R_k'$, $e^{2\pi s_k} := R_k$. Then, the stated result for $\tilde{v}$ implies the result for $\tilde{u}$.

2. $\exists c > 0 : \forall (s, t) \in \mathbb{R} \times S^1 : |\nabla \tilde{v}(s, t)| \leq c$.

   Proof: By exercise 6.1.

3. For each $k \in \mathbb{N}$, define $\tilde{v}_k = (b_k, v_k) : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times M$ by
   $$\tilde{v}_k(s, t) = (b(s + s_k, t) - b(s_k, 0), v(s_k + s, t)).$$

4. $\tilde{v}_k$ satisfies
   (i) $\tilde{v}_k$ is holomorphic on $\mathbb{R} \times S^1$,
   (ii) $0 < E(\tilde{v}_k) < +\infty$,
   (iii) $\exists p \in M : \forall t \in S^1 : \lim_{s \to -\infty} v_k(s, t) = p$,
   (iv) $b_k(0, 0) = 0$.

   Proof: (i), (ii), (iii) follow from step 1 (i), (ii), (iii). (iv) follows from the definition of b_k.

5. There exists a subsequence of s_k, (whose index we still denote by k) and $\tilde{w} = (\beta, w) \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times M)$ such that $\tilde{v}_k$ converges in the C^∞_W-topology to $\tilde{w}$.

   Proof: Similar to the one in exercise 5.2. By step 4, the C^∞-bounds theorem gives bounds on higher derivatives coming from the gradient bounds. Then apply the Arzelà-Ascoli theorem as in exercise 5.2.

6. $\tilde{w}$ satisfies
   (i) $\tilde{w}$ is holomorphic on $\mathbb{R} \times S^1$,
   (ii) $0 < E(\tilde{w}) < +\infty$,
   (iii) $\forall s_0 \in \mathbb{R} : \int_{\{s_0\} \times S^1} w^* \lambda = T$,
   (iv) $\int_{\mathbb{R} \times S^1} w^* d\lambda = 0$.

Proof:

6.1. (i)

Proof: By steps 4 (i) and 5.

6.2. (ii)

Proof: By steps 4 (ii) and 5.

6.3. (iii)

Proof:

$$\begin{aligned}
\int_{\{s_0\}\times S^1} w^*\lambda &= \lim_{k\to+\infty}\int_{\{s_0\}\times S^1} v_k^*\lambda && \text{[step 5]}\\
&= \lim_{k\to+\infty}\int_{(-\infty,s_0+s_k]\times S^1} v^*d\lambda && \text{[Stokes + change var.]}\\
&= \int_{\mathbb{R}\times S^1} v^*d\lambda\\
&= T && \text{[assumption in step 1 (iv)].}
\end{aligned}$$

6.4. (iv)

Proof:

$$\begin{aligned}
\int_{\mathbb{R}\times S^1} w^*d\lambda &= \lim_{R\to+\infty}\int_{[-R,R]\times S^1} w^*d\lambda\\
&= \lim_{R\to+\infty}\lim_{k\to+\infty}\int_{[-R,R]\times S^1} v_k^*d\lambda && \text{[step 5]}\\
&= \lim_{R\to+\infty}\lim_{k\to+\infty}\int_{[-R+s_k,R+s_k]\times S^1} v^*d\lambda && \text{[change var.]}\\
&= \lim_{R\to+\infty} 0 && \left[\textstyle\int_{\mathbb{R}\times S^1} v^*d\lambda<+\infty\right]\\
&= 0.
\end{aligned}$$

7. Consider the periodic extension of $\tilde{w}\colon \mathbb{R}\times S^1 \longrightarrow \mathbb{R}\times M$ to $\mathbb{C}\longrightarrow \mathbb{R}\times M$, and denote it by $\tilde{w}$ as well.

8. There exists $f\in C^\infty(\mathbb{C},\mathbb{R})$ such that $df = w^*d\lambda$.

   Proof: Since $\int_{\mathbb{R}\times S^1} w^*d\lambda = 0$, then $\int_{\mathbb{C}} w^*d\lambda = 0$. Use the same argument as in exercise 5.2.

9. Let $\Phi = \beta + if\colon \mathbb{C}\longrightarrow\mathbb{C}$. Then, Φ

   (i) is holomorphic,

   (ii) is non-constant,

   (iii) has bounded gradient.

   Proof:

   9.1. (i)

   Proof: By $df = w^*d\lambda$ and $\tilde{w}$ being holomorphic, Φ satisfies the Cauchy-Riemann equations.

   9.2. (ii)

   Proof: Assume by contradiction that Φ is constant. Then, $0 = d\beta = (w^*\lambda)\circ i$, so $w^*\lambda = 0$. Since $\int_{\mathbb{C}} w^*d\lambda = 0$ and the integrand is positive, $w^*d\lambda = 0$. By $TM = \ker\lambda\oplus\ker d\lambda$, $w = 0$. Contradiction.

9.3. (iii)

Proof:

$$\begin{aligned}\sup_{z\in\mathbb{C}}|\nabla\Phi|^2 &= 2\sup_{z\in\mathbb{C}}|\nabla\beta|^2 && [\Phi = \beta + if \text{ is holomorphic}]\\ &\leq 2\sup_{z\in\mathbb{C}}|\nabla\tilde{w}|^2 && [\tilde{w} = (\beta, w), \text{ triangular ineq.}]\\ &< +\infty,\end{aligned}$$

where the last step is true because step 2 implies a gradient bound for v_k, which implies a gradient bound for w.

10. Φ is of the form $\Phi(z) = az + b$, for $a = a_1 + ia_2, b = b_1 + ib_2 \in \mathbb{C}, a \neq 0$.

Proof: Step 9 and Liouville's theorem.

11. $\beta(s + it) = a_1 s + b_1$.

Proof:

$$\begin{aligned}\beta(s+it) &= \mathrm{Re}(\Phi(s+it))\\ &= a_1 s - a_2 t + b_1.\end{aligned}$$

Since β is 1-periodic in t, $a_2 = 0$.

12. For each s, $w_t = a_1 X_\lambda(w)$ and $x(t) \coloneqq w(s, a_1^{-1}t)$ is an orbit of the Reeb vector field of period 1.

Proof:

12.1. $w_s = 0$.

Proof:

$$\begin{aligned}w_s &= \pi_\lambda w_s + \lambda(w_s)X_\lambda(w) && [TM = \ker\lambda \oplus \ker d\lambda]\\ &= \lambda(w_s)X_\lambda(w) && [\textstyle\int_{\mathbb{C}} w^* d\lambda = 0]\\ &= -\beta_t X_\lambda(w) && [d\beta = (w^*\lambda)\circ i]\\ &= 0 && [\text{step } 11].\end{aligned}$$

12.2. $w_t = a_1 X_\lambda(w)$.

Proof:

$$\begin{aligned}w_t &= \pi_\lambda w_t + \lambda(w_t)X_\lambda(w) && [TM = \ker\lambda \oplus \ker d\lambda]\\ &= \lambda(w_t)X_\lambda(w) && [\textstyle\int_{\mathbb{C}} w^* d\lambda = 0]\\ &= -\beta_s X_\lambda(w) && [d\beta = (w^*\lambda)\circ i]\\ &= a_1 X_\lambda(w) && [\text{step } 11].\end{aligned}$$

12.3. $\partial_t w(s, a_1^{-1}t) = X_\lambda(w(s, a_1^{-1}t))$.

Proof: By step 12.2 and the chain rule.

12.4. Q.E.D.

Proof: Steps 12.2, 12.3 and the definition of Reeb vector field.

13. $\tilde{w}(s,t) = (Ts + b, x(Tt))$.

Proof:

13.1. $T = a_1$.

Proof:

$$\begin{aligned} T &= \int_{\{s_0\}\times S^1} w^*\lambda && \text{[step 6 (iii)]} \\ &= \int_0^1 \lambda(w_t)dt && \text{[write the integrand in coordinates]} \\ &= \int_0^1 a_1 dt && \text{[step 12]} \\ &= a_1. \end{aligned}$$

13.2. Q.E.D.

Proof:

$$\begin{aligned} \tilde{w}(s,t) &= (\beta(s,t), w(s,t)) \\ &= (a_1 s + b, x(a_1 t)) && \text{[steps 11 and 12]} \\ &= (Ts + b_1, x(Tt)) && \text{[step 13.1].} \end{aligned}$$

14. Q.E.D.

Proof:

$$\begin{aligned} \overset{C^\infty}{\lim_{k\to+\infty}} v(s_k,\cdot) &= \overset{C^\infty}{\lim_{k\to+\infty}} v_k(0,\cdot) && \text{[step 3]} \\ &= w(0,\cdot) && \text{[step 5]} \\ &= x(T\cdot) && \text{[step 13].} \end{aligned}$$

□

7 Exercise sheet No. 7 - 25-06-2019

In this exercise sheet we are going to prove the spectral theorem for compact, self adjoint operators:

Theorem (Spectral theorem for compact self-adjoint operators)**.** *Let H be a complex Hilbert space and $T\colon H \longrightarrow H$ be a compact self-adjoint operator. Let $r = \operatorname{rank}(T) = \dim(\operatorname{im} T)$. Then, there exist sets $\{\lambda_n\}_{n=1}^r$, $\{e_n\}_{n=1}^r$ of eigenvectors and corresponding eigenvalues such that:*

(i) *$\{e_n\}_{n=1}^r$ is an orthonormal basis for $\overline{\operatorname{im} T}$;*

(ii) *$\{\lambda_n\}_{n=1}^r$ is the set of nonzero eigenvalues of T;*

(iii) *$|\lambda_1| \geq |\lambda_2| \geq \ldots > 0$;*

(iv) *If $r = \infty$ then $\lim_{n\to\infty} \lambda_n = 0$;*

(v) *$\forall x \in H\colon Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n$.*

The proof we give is based on [RY00]. For hints on how to solve the exercises, read the proof sketches in the solutions. Let H be a Hilbert space and T be a compact, self-adjoint bounded operator.

Exercise 7.1. Show that
$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Proof sketch: ($\geq$) is a short computation that uses the Cauchy-Schwarz inequality. For ($\leq$), show that
$$\forall y, z \in H\colon 4\operatorname{Re}\langle Ty, z\rangle \leq 2 \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$
The proof of this last equality uses the parallelogram law. Then for each x with norm 1 find y, z such that $\|Tx\| = 2\operatorname{Re}\langle Ty, z\rangle$.
Solution:
1. ($\geq$) :

   Proof: By definition of supremum, it suffices to assume that $x \in H$, $\|x\| = 1$, and prove that $\|T\| \geq |\langle Tx, x\rangle|$.
$$\begin{aligned} \|T\| &\geq \|Tx\| && \text{[def. of norm of operator]} \\ &= \|Tx\|\|x\| && [\|x\| = 1] \\ &\geq |\langle Tx, x\rangle| && \text{[Cauchy-Schwarz inequality].} \end{aligned}$$

2. ($\leq$) :

   Proof:

   2.1. It suffices to assume that $x \in H$, $\|x\| = 1$ and prove $\|Tx\| \leq \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

   2.2. $\forall y, z \in H\colon 4\operatorname{Re}\langle Ty, z\rangle \leq 2\sup_{\|x\|=1} |\langle Tx, x\rangle|$.

   Proof:
$$\begin{aligned} &4\operatorname{Re}\langle Ty, z\rangle \\ &\quad= 2(\langle Ty, z\rangle + \overline{\langle Ty, z\rangle}) && [\forall \tau \in \mathbb{C}\colon 2\operatorname{Re}\tau = \tau + \bar{\tau}] \end{aligned}$$

$$
\begin{aligned}
&= 2(\langle Ty, z\rangle + \langle z, Ty\rangle) && \text{[def. of complex Hilbert space]}\\
&= 2(\langle Ty, z\rangle + \langle Tz, y\rangle) && \text{[T is self adjoint]}\\
&= \langle T(y+z), y+z\rangle - \langle T(y-z), y-z\rangle && \text{[algebra]}\\
&\le |\langle T(y+z), y+z\rangle| - |\langle T(y-z), y-z\rangle|\\
&\le 2 \sup_{\|x\|=1} |\langle Tx, x\rangle| \big(\|y+z\|^2 + \|y-z\|^2\big)\\
&= 2 \sup_{\|x\|=1} |\langle Tx, x\rangle| \big(\|y\|^2 + \|z\|^2\big) && \text{[paralelogram law].}
\end{aligned}
$$

2.3. $\exists y, z \in H\colon 2\operatorname{Re}\langle Ty, z\rangle = \|Tx\|$.

Proof: Let $y = x$ and $z = \frac{Tx}{2\|Tx\|}$. Then,

$$
\begin{aligned}
2\operatorname{Re}\langle Ty, z\rangle &= 2\operatorname{Re}\left\langle Tx, \frac{Tx}{2\|Tx\|^2}\right\rangle\\
&= 2\operatorname{Re}\frac{1}{2\|Tx\|}\|Tx\|^2\\
&= \|Tx\|.
\end{aligned}
$$

2.4. Q.E.D.

Proof: By steps 2.2 and 2.3, $\|Tx\| = 2\operatorname{Re}\langle Ty, z\rangle \le \sup_{\|x\|=1} |\langle Tx, x\rangle|$. By step 2.1 this proves the result.

3. Q.E.D.

Proof: Steps 1 and 2. □

Exercise 7.2. Show that $\|T\|$ is an eigenvalue of T or $-\|T\|$ is an eigenvalue of T.

Proof sketch: Use exercise 7.1 to create a sequence x_n such that $\langle Tx_n, x_n\rangle$ converges to $\lambda = \pm\|T\|$. Since T is compact, some subsequence of Tx_n converges to y. Use the two established limits

$$
\lim_{n\to\infty} \langle Tx_n, x_n\rangle = \lambda,
$$
$$
\lim_{n\to\infty} Tx_n = y,
$$

to show that $Ty = \lambda y$.

Solution:

1. There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ such that $\|x_n\| = 1$ for each n and such that $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$, where $\lambda = \|T\|$ or $\lambda = -\|T\|$.

   Proof: By exercise 7.1, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ such that $\|x_n\| = 1$ for each n and $\lim_{n\to\infty} |\langle Tx_n, x_n\rangle| = \|T\|$. Consider the subsets of $\mathbb{N}$

   $$
   S^+ = \{n \in \mathbb{N} \mid \langle Tx_n, x_n\rangle > 0\},
   $$
   $$
   S^- = \{n \in \mathbb{N} \mid \langle Tx_n, x_n\rangle < 0\}.
   $$

   At least one of these subsets is infinite. Take the subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that corresponds to that subset. If that subsequence is S^+ then

   $$
   \lim_{n\to\infty} \langle Tx_n, x_n\rangle = \|T\|
   $$

   and if it is S^- then

   $$
   \lim_{n\to\infty} \langle Tx_n, x_n\rangle = -\|T\|.
   $$

2. There exists a subsequence of x_n, whose index we still denote by n, and $y \in H$ such that Tx_n converges to y.

Proof: T is compact and $\|x_n\| = 1$.

3. $\lim_{n\to\infty} \|Tx_n - \lambda x_n\|^2 = 0$.

   Proof:
$$\begin{aligned}\lim_{n\to\infty} \|Tx_n - \lambda x_n\|^2 &= \lim_{n\to\infty}\Big(\|Tx_n\|^2 + \lambda^2 - 2\lambda\langle Tx_n, x_n\rangle\Big) \\ &\leq \lim_{n\to\infty}\Big(2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle\Big) && [\lambda^2 = \|T\|^2] \\ &= 0 && [\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda].\end{aligned}$$

4. $\lim_{n\to\infty} x_n = \frac{1}{\lambda} y$.

   Proof:
$$\begin{aligned}y &= \lim_{n\to\infty} Tx_n && [\text{step 3}] \\ &= \lim_{n\to\infty} Tx_n + \lim_{n\to\infty}(\lambda x_n - Tx_n) && [\text{step 2}] \\ &= \lim_{n\to\infty} Tx_n\end{aligned}$$

5. $Ty = \lambda y$.

   Proof:
$$\begin{aligned}\lambda y &= \lim_{n\to\infty} \lambda T x_n && [\text{step 3}] \\ &= \lambda T\Big(\lim_{n\to\infty} x_n\Big) && [T \text{ is continuous}] \\ &= \lambda T \frac{1}{\lambda} y && [\text{step 4}] \\ &= Ty.\end{aligned}$$

6. Q.E.D.

   Proof: By step 5 λ is an eigenvalue of T, where by step 1 $\lambda = \pm\|T\|$. □

Exercise 7.3. Show that for all $t > 0$ the set
$$\{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } T, |\lambda| \geq t\}$$
is finite.

Proof sketch: Assume by contradiction that there exists $t_0 > 0$ and a sequence of pairwise distinct eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $|\lambda_n| > t_0$. Let $\{e_n\}_{n\in\mathbb{N}}$ be a sequence of associated unit eigenvectors. Since T is self-adjoint the e_n are an orthonormal set. Show that $\|Te_n - Te_m\| \geq |t_0|$ for all n, m distinct. This contradicts compactness of T.
Solution:

1. Assume by contradiction that there exists $t_0 > 0$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of pairwise distinct eigenvalues of T with $|\lambda_n| \geq t_0$ and a corresponding sequence $\{e_n\}_{n\in\mathbb{N}}$ of unit eigenvectors.

2. $\forall n, m \in \mathbb{N}$: $\langle e_n, e_m\rangle = \delta_{nm}$.

   Proof: If $n = m$ then $\langle e_n, e_m\rangle = 1$ because the vectors have unit norm. If $n \neq m$, then $\lambda_n \neq \lambda_m$ and
$$\begin{aligned}\lambda_n\langle e_n, e_m\rangle &= \langle Te_n, e_m\rangle \\ &= \langle e_n, Te_m\rangle \\ &= \langle e_n, e_m\rangle\lambda_m \\ \implies 0 &= \langle e_n, e_m\rangle.\end{aligned}$$

3. $\forall n, m \in \mathbb{N}$ with $n > m$: $\|Te_n - Te_m\| \geq |t_0|$.

   Proof:
$$\begin{aligned}\|Te_n - Te_m\|^2 &= \|\lambda_n e_n - \lambda_m e_m\|^2 \\ &= \langle \lambda_n e_n - \lambda_m e_m, \lambda_n e_n - \lambda_m e_m \rangle \\ &= \|\lambda_n e_n\|^2 - 2\langle \lambda_m e_m, \lambda_n e_n \rangle + \|\lambda_m e_m\|^2 \\ &= |\lambda_n|^2 + |\lambda_m|^2 && \text{[step 2]} \\ &\geq 2|t_0|^2.\end{aligned}$$

4. Q.E.D.

   Proof: By step 3, Te_n does not have a convergent subsequence. By compactness of T, Te_n has a convergent subsequence. Contradiction. □

By exercise 7.3, the set of eigenvalues of T is countable. By compactness of T, each eigenvalue λ has finite multiplicity m_λ. There exists a list $\{\lambda_n\}_{n=1}^\infty$ containing all the eigenvalues of T, that is ordered from biggest to lowest in absolute value, such that each eigenvalue λ shows up in the list m_λ times. For each eigenvalue λ in this list, by the Gram-Schmidt algorithm there exists an orthonormal basis of $\ker(T - I\lambda)$ consisting of m_λ eigenvectors. Consider the list $\{e_n\}_{n=1}^\infty$ obtained from listing the eigenvectors obtained previously in the same order as the eigenvalues. Consider the truncated list $\{\lambda_n\}_{n=1}^J$ that only has nonzero eigenvalues and the corresponding list $\{e_n\}_{n=1}^J$.

Exercise 7.4. Show that $\{e_n\}_{n=1}^J$ is an orthonormal set.

Proof sketch: If the eigenvectors are in the same eigenspace, then the result follows by the construction of the eigenvectors. If they are not, show that $\lambda_n\langle e_n, e_m\rangle = \langle e_n, e_m\rangle\lambda_m$ using the fact that T is self adjoint.
Solution:

1. It suffices to assume that $n, m = 1, \dots, J$ and prove that $\langle e_n, e_m \rangle = \delta_{nm}$.
2. Case $\lambda_n = \lambda_m$.

   Proof: Then e_n, e_m correspond to the same eigenvalue $\lambda_n = \lambda_m$, so they are part of an orthonormal basis for $\ker(T - \lambda_n I)$ by construction. Therefore $\langle e_n, e_m \rangle = \delta_{nm}$.
3. Case $\lambda_n \neq \lambda_m$.

   Proof:
$$\begin{aligned}\lambda_n \langle e_n, e_m \rangle &= \langle \lambda_n e_n, e_m \rangle \\ &= \langle Te_n, e_m \rangle \\ &= \langle e_n, Te_m \rangle && \text{[T is self adjoint]} \\ &= \langle e_n, \lambda_m e_m \rangle \\ &= \langle e_n, e_m \rangle \lambda_m \\ \implies 0 &= \langle e_n, e_m \rangle && [\lambda_n \neq \lambda_m].\end{aligned}$$
□

Exercise 7.5. Show that $\overline{\operatorname{span}\{e_n\}_{n=1}^J} = \overline{\operatorname{im} T}$.

Proof sketch: Let $M := \overline{\operatorname{span}\{e_n\}_{n=1}^J}$. Using the fact that the e_n are eigenvectors of T, it's easy to show that $M \subset \overline{\operatorname{im} T}$. $M = \overline{\operatorname{im} T}$ is equivalent to $M^\perp = (\overline{\operatorname{im} T})^\perp$, which is what we prove. Using the properties of the orthogonal complement and T being self-adjoint, we

can show that $(\overline{\operatorname{im} T})^\perp = \ker T$. Then, $\ker T = (\overline{\operatorname{im} T})^\perp \subset M^\perp$ follows from $M \subset \overline{\operatorname{im} T}$. For $M^\perp \subset \ker T$, since M is the span of the eigenvectors of T it is invariant under T. Therefore, since T is self-adjoint, $M^\perp$ is also invariant under T. Therefore it suffices to show that the map $T|_{M^\perp} \colon M \longrightarrow M$ is zero. Assume that it is nonzero. Then by exercise 7.2, it has a nonzero eigenvalue, which is also an eigenvalue of T. The corresponding eigenvector is in $M^\perp$ because it is an eigenvector of $T|_{M^\perp}$ but it is also in $M^\perp$ because $M = \overline{\{e_n\}_{n=1}^J}$ contains all the eigenvalues associated to nonzero eigenvectors. Then the eigenvector is zero, which is a contradiction and therefore $T|_{M^\perp} = 0$.
Solution:

1. Let $M \coloneqq \overline{\operatorname{span}\{e_n\}_{n=1}^J}$.
2. $M \subset \overline{\operatorname{im} T}$.

   Proof:

   2.1. It suffices to assume that $u \in M$ and prove that $u \in \overline{\operatorname{im} T}$.

   2.2. $u = \sum_{n=1}^J \langle u, e_n\rangle e_n$.

   Proof: Since $u \in \overline{\operatorname{span}\{e_n\}_{n=1}^J}$.

   2.3. Case $J < +\infty$.

   Proof:

$$
\begin{aligned}
u &= \sum_{n=1}^J \langle u, e_n\rangle e_n \\
&= \sum_{n=1}^J \langle u, e_n\rangle \frac{1}{\lambda_n} T e_n \\
&= T\left(\sum_{n=1}^J \langle u, e_n\rangle \frac{1}{\lambda_n} e_n\right).
\end{aligned}
$$

   2.4. Case $J = \infty$.

   Proof:

$$
\begin{aligned}
u &= \lim_{k\to\infty} \sum_{n=1}^k \langle u, e_n\rangle e_n \\
&= \lim_{k\to\infty} \sum_{n=1}^k \langle u, e_n\rangle \frac{1}{\lambda_n} T e_n \\
&= T\left(\sum_{n=1}^J \langle u, e_n\rangle \frac{1}{\lambda_n} e_n\right) \qquad [T \text{ is continuous}].
\end{aligned}
$$

   2.5. Q.E.D.

   Proof: Steps 2.1, 2.3 and 2.4.

3. $\ker T = M^\perp$.

   Proof:

   3.1. $\ker T \subset M^\perp$.

   Proof:

$$
\begin{aligned}
\ker T &= (\operatorname{im} T^*)^\perp && [\text{general fact of operators on Hilbert spaces}] \\
&= (\operatorname{im} T)^\perp && [T \text{ is self adjoint}]
\end{aligned}
$$

$$
\begin{aligned}
&= (\operatorname{im} T)^{\perp\perp\perp} && \text{[general fact of orthogonal complements]}\\
&= (\overline{\operatorname{im} T})^{\perp} && \text{[general fact of orthogonal complements]}\\
&\subset M^{\perp} && \text{[Step 3 and general fact of orthogonal complements].}
\end{aligned}
$$

3.2. $M^\perp \subset \ker T$.

Proof:

3.2.1. $T(M) \subset M$.

Proof:

3.2.1.1. It suffices to assume that $u \in M$ and prove that $Tu \in M$.

3.2.1.2. $u = \sum_{n=1}^{J} \langle u, e_n \rangle e_n$.

Proof: Since $u \in \overline{\operatorname{span}\{e_n\}_{n=1}^{J}}$.

3.2.1.3. Case $J < \infty$.

Proof:

$$
\begin{aligned}
Tu &= T\left(\sum_{n=1}^{J} \langle u, e_n \rangle e_n\right) && \text{[Step 3.2.1.2]}\\
&= \sum_{n=1}^{J} \langle u, e_n \rangle T e_n\\
&= \sum_{n=1}^{J} \langle u, e_n \rangle \lambda_n e_n\\
&\in M.
\end{aligned}
$$

3.2.1.4. Case $J = \infty$.

Proof:

$$
\begin{aligned}
Tu &= T\left(\lim_{k\to\infty} \sum_{n=1}^{J} \langle u, e_n \rangle e_n\right) && \text{[Step 3.2.1.2]}\\
&= \lim_{k\to\infty} \sum_{n=1}^{J} \langle u, e_n \rangle T e_n && \text{[T is continuous]}\\
&= \lim_{k\to\infty} \sum_{n=1}^{J} \langle u, e_n \rangle \lambda_n e_n\\
&\in M.
\end{aligned}
$$

3.2.1.5. Q.E.D.

Proof: Step 3.2.1.1, 3.2.1.3 and 3.2.1.4.

3.2.2. Let $N = M^\perp$. Then N is invariant under T. Let $T_N \colon N \longrightarrow N$ be the restriction of T to N.

Proof:

3.2.2.1. It suffices to assume that $v \in M^\perp$ and prove that $Tv \in M^\perp$.

3.2.2.2. It suffices to assume that $\forall u \in M \colon \langle v, u \rangle = 0$ and prove that $\forall w \in M \colon \langle Tv, w \rangle = 0$.

3.2.2.3. $Tw \in M$.

Proof: $w \in M$ and M is invariant under T.

3.2.2.4. Q.E.D.

Proof:

$$\begin{aligned} \langle Tv, w\rangle &= \langle v, Tw\rangle && [T \text{ is self adjoint}] \\ &= 0 && [\text{By hypothesis}]. \end{aligned}$$

3.2.3. T_N is a compact self-adjoint operator.

Proof: It is the restriction of a compact self-adjoint operator.

3.2.4. $T_N = 0$.

Proof:

3.2.4.1. Assume by contradiction that $T_N \neq 0$.

3.2.4.2. T_N has a nonzero eigenvalue $\tilde{\lambda}$ with corresponding eigenvector $\tilde{e} \in N$ nonzero.

Proof: Exercise 7.2.

3.2.4.3. $\tilde{\lambda}$ is an eigenvalue of T with corresponding eigenvector $\tilde{e}$.

Proof:

$$\begin{aligned} T\tilde{e} &= T_N\tilde{e} && [\tilde{e} \in N] \\ &= \tilde{\lambda}\tilde{e}. \end{aligned}$$

3.2.4.4. For some $n = 1, \ldots, J$, $\tilde{\lambda} = \lambda_n$ and $\tilde{e} = e_n$.

Proof: By steps 3.2.4.2 and 3.2.4.3 and the fact that the list $\{\lambda_n\}_{n=1}^{J}$ contains all nonzero eigenvalues.

3.2.4.5. Q.E.D.

Proof: Since

$$\begin{aligned} M^{\perp} &\ni \tilde{e} && [\text{Step 3.2.4.2}] \\ &= e_n && [\text{Step 3.2.4.4}] \\ &\in M && [\text{def. of } M], \end{aligned}$$

$\tilde{e} = 0$, but since $\tilde{e}$ is an eigenvalue $\tilde{e} \neq 0$. Contradiction.

3.2.5. Q.E.D.

Proof: Since $T_N = 0$, then $\forall v \in N\colon Tv = 0$, i.e. $v \in \ker T$.

3.3. Q.E.D.

Proof: Steps 3.1 and 3.2.

4. Q.E.D.

Proof:

$$\begin{aligned} \overline{\operatorname{im} T} &= (\ker T)^{\perp} && [T \text{ is self-adjoint}] \\ &= M^{\perp\perp} && [\text{Step 3}] \\ &= M && [M \text{ is closed}]. \end{aligned}$$

□

Exercise 7.6. Show that $J = r$.

Proof sketch: This is simply because $J = \dim(\overline{\operatorname{im} T})$ by exercise 7.5 and $r = \dim(\operatorname{im} T)$ by the definition of rank. If J and r are finite, then $\operatorname{im} T = \overline{\operatorname{im} T}$. If not, notice that $\dim \operatorname{im} T \leq \dim \overline{\operatorname{im} T}$ and that J is countably infinite.

Solution:

1. Case J finite.

   Proof:
   $$\begin{aligned} J &= \dim(\overline{\operatorname{im} T}) && \text{[exercise 7.5]} \\ &= \dim(\operatorname{im} T) && \text{[if } \dim(\overline{\operatorname{im} T}) \text{ is finite then } \overline{\operatorname{im} T} = \operatorname{im} T] \\ &= r && \text{[def. of rank]} \end{aligned}$$

2. Case J countably infinite.

   Proof:
   2.1. $r \leq J$.

   Proof:
   $$\begin{aligned} J &= \dim(\overline{\operatorname{im} T}) && \text{[exercise 7.5]} \\ &\geq \dim(\operatorname{im} T) && [\operatorname{im} T \subset \overline{\operatorname{im} T}] \\ &= r && \text{[def. of rank]} \end{aligned}$$

   2.2. $r \geq J$.

   Proof: Assume by contradiction that $r < J$, in other words that r is finite. Then $\operatorname{im} T$ is finite dimensional, so $\operatorname{im} T = \overline{\operatorname{im} T}$. So $r = J$. Contradiction.

   2.3. Q.E.D.

   Proof: Steps 2.1 and 2.2

3. Q.E.D.

   Proof: Steps 1 and 2. □

Exercise 7.7. Prove the spectral theorem.

Solution:

1. (i):

   Proof: Exercises 7.4, 7.5 and 7.6.

2. (ii):

   Proof: By construction of the set $\{\lambda_n\}_{n=1}^J$ and $J = r$.

3. (iii):

   Proof: By construction of the set $\{\lambda_n\}_{n=1}^J$ and $J = r$.

4. (iv):

   Proof sketch: By exercise 7.3.
   Proof: Assume by contradiction that $J = \infty$ and that λ_n does not converge to 0. There exists $\varepsilon > 0$ and a subsequence of λ_n, whose index we still denote by n, such that $|\lambda_n| \geq \varepsilon$. By exercise 7.3, this is a contradiction.

5. (v):

   Proof sketch: Consider $P \colon H \longrightarrow M$ the orthogonal projection. Since $Px \in M$, we can expand it in a basis:

$$Px = \sum_{n=1}^{J} \langle Px, e_n \rangle e_n.$$

The result follows from computing $Tx = TPx$ with the basis expansion for Px.

Proof: Let P denote the orthogonal projection to M.

$$\begin{aligned}
Tx &= T(Px + (I-P)x) && [H = M \oplus M^\perp] \\
&= TPx && [I-P \text{ is the projection onto } M^\perp = \ker T] \\
&= T\left(\sum_{n=1}^{J} \langle Px, e_n \rangle e_n\right) && [Px = M = \overline{\{e_n\}_{n=1}^{J}}] \\
&= \sum_{n=1}^{J} \langle Px, e_n \rangle Te_n && [T \text{ is linear bounded}] \\
&= \sum_{n=1}^{J} \lambda_n \langle Px, e_n \rangle e_n && [Te_n = \lambda_n e_n] \\
&= \sum_{n=1}^{J} \lambda_n \langle x, e_n \rangle e_n && [(I-P)x \in M^\perp \text{ and } e_n \in M \implies \langle (I-P)x, e_n \rangle = 0].
\end{aligned}$$

6. Q.E.D.

   Proof: Steps 1, 2, 3, 4 and 5. □

8 Exercise sheet No. 8 - 26-06-2019

In this exercise sheet we are going to prove the Carleman similarity principle:

Theorem (Carleman similarity principle)**.** *Let $\varepsilon > 0$, $p > 2$, $C \in L^p(B_\varepsilon, \mathrm{Hom}_\mathbb{R}(\mathbb{C}^n, \mathbb{C}^n))$, $J \in W^{1,p}(B_\varepsilon, \mathrm{Hom}_\mathbb{R}(\mathbb{C}^n, \mathbb{C}^n))$ such that $\forall z \in B_\varepsilon \colon J(z)^2 = -\mathrm{id}_{\mathbb{C}^n}$, and $u \in W^{1,p}(B_\varepsilon, \mathbb{C}^n)$ be such that $\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0$ and $u(0) = 0$. Then, there exist $\delta \in (0, \varepsilon)$ and $\Phi \in W^{1,p}(B_\delta, \mathrm{Hom}_\mathbb{R}(\mathbb{C}^n, \mathbb{C}^n))$ such that:*

(i) $\forall z \in B_\delta \colon \Phi(z)$ *is invertible;*

(ii) $\forall z \in B_\delta \colon J(z)\Phi(z) = \Phi(z)i$;

(iii) *The map*

$$\begin{aligned} \sigma \colon B_\delta &\longrightarrow \mathbb{C}^n \\ z &\longmapsto \Phi(z)^{-1}u(z) \end{aligned}$$

is holomorphic.

Let V be a complex finite dimensional vector space and (S^2, j) be the sphere with its standard complex structure. Let

$$\begin{aligned} \bar{\partial} \colon W^{1,p}(S^2, V) &\longrightarrow L^p(S^2, \overline{\mathrm{Hom}}_\mathbb{C}(TS^2, V)) \\ u &\longmapsto du + i \circ du \circ j. \end{aligned}$$

The proof that we give is based on [MS04] and [FHS95].To prove the Carleman similarity principle, we will need the following theorem. It is a particular case of the Riemann-Roch theorem, which is a theorem stating the facts about differential operators coming from Fredholm theory. We present an informal proof sketch of it, based on the explanation given in [FHS95], for the sake of completeness.

Theorem (Riemann-Roch for $\bar{\partial}$)**.** *For every $p > 1$,*

(i) $\bar{\partial}$ *is a Fredholm operator;*

(ii) $\ker \bar{\partial} = \{u \in W^{1,p}(S^2, V) \mid u \text{ is constant}\}$;

(iii) $\bar{\partial}$ *is surjective;*

(iv) *As an operator between complex Banach spaces,* $\mathrm{ind}\, \bar{\partial} = \dim_\mathbb{C} V$.

Proof sketch:

1. (i):

   Proof sketch: Use Weyl's lemma (the lemma that ensures semi-Fredholm provided a certain inequality holds) and the Calderon-Zygmund inequality to prove the necessary inequality.

2. (ii):

   Proof sketch: Let $u \in \ker \bar{\partial}$. By elliptic regularity, u is C^∞ and therefore holomorphic. Since u is defined on S^2, it is bounded. By Liouville's theorem, u is constant.

3. (iii):

   Proof sketch: We will explain how one could define a map
   $$R\colon L^p(S^2, \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, V)) \longrightarrow W^{1,p}(S^2, V)$$
   which is a right inverse of $\bar{\partial}$. Consider the North and South pole in S^2, $N, S \in S^2$, and the associated stereographic projections $\phi_N\colon S^2\backslash\{N\} \longrightarrow \mathbb{C}$, $\phi_S\colon S^2\backslash\{S\} \longrightarrow \mathbb{C}$. Using ϕ_N, $R|_{S^2\backslash\{N\}}$ can be viewed as a map
   $$R|_{S^2\backslash\{N\}}\colon L^p(\mathbb{C}, \overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}, V)) \longrightarrow W^{1,p}(\mathbb{C}, V),$$
   and analogously for $R|_{S^2\backslash\{S\}}$. To define R, it suffices to define it on $S^2\backslash\{N\}$ and on $S^2\backslash\{S\}$. Let
   $$T\colon C_0^\infty(\mathbb{C}, V) \longrightarrow C^\infty(\mathbb{C}, V)$$
   be given by
   $$(Tv)(z) = \lim_{\varepsilon\to 0}\left(-\frac{1}{2\pi}\int_{\mathbb{C}\backslash B_\varepsilon(0)} \frac{v(z+\zeta)}{\zeta} d\zeta\right).$$
   Define $R|_{S^2\backslash\{N\}}$ by
   $$\begin{aligned} R|_{S^2\backslash\{N\}}\colon C_0^\infty(\mathbb{C}, \overline{\mathrm{Hom}}_{\mathbb{C}}(\mathbb{C}, V)) &\longrightarrow C^\infty(\mathbb{C}, V) \\ v d\bar{z} &\longmapsto Tv, \end{aligned}$$
   and $R|_{S^2\backslash\{S\}}$ analogously (by the same formula as above, but it would be imprecise to say that they are defined equally since in each case there are different identifications happening). $R|_{S^2\backslash\{N\}}$ and $R|_{S^2\backslash\{S\}}$ extend to operators from L^p to $W^{1,p}$ that agree on the intersection of their domains, so we have a map
   $$R\colon L^p(S^2, \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, V)) \longrightarrow W^{1,p}(S^2, V).$$
   This map is such that
   $$\bar{\partial}\circ R = \mathrm{id}_{L^p(S^2, \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, V))}.$$

4. (iv):

   Proof sketch: By steps 2 and 3. □

Exercise 8.1. Show that we may assume that $J(z) = i$.

Proof:

1. It suffices to assume that the theorem holds in the case $J(z) = i$ and the hypothesis of the theorem, and to prove that the conclusions of the theorem are true.
2. Let $\Psi \in W^{1,p}(B_\varepsilon, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}^n))$ be such that for each $z \in B_\varepsilon$ $\Psi(z)$ is invertible and $\Psi(z)^{-1}J(z)\Psi(z) = i$.

   Proof: Let $u_1, \ldots, u_n$ be the canonical basis of $\mathbb{R}^n$. Then, $u_1, J(z)u_1, \ldots, u_n, J(z)u_n$ is a basis of $\mathbb{C}^n$ as a vector space over $\mathbb{R}$. Define $\Psi(z)$ by
   $$\Psi(z)(x_1 + iy_1, \ldots, x_n + iy_n) = \sum_{j=1}^n (x_j u_j + y_j J(z)u_j).$$
   Then Ψ is the desired map.
3. Let $v \in W^{1,p}(B_\varepsilon, \mathbb{C}^n)$ be given by $v(z) = \Psi(z)^{-1}u(z)$.
4. $\partial_s v + i\partial_t v + \tilde{C}v = 0$, where $\tilde{C} \coloneqq \Psi^{-1}(\partial_s\Psi + J\partial_t\Psi + C\Psi) \in L^p(B_\varepsilon, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C},^n))$.

   Proof:
   $$\begin{aligned} 0 &= \partial_s u + J\partial_t u + Cu && \text{[by hypothesis]} \\ &= \partial_s(\Psi v) + i\partial_t(\Psi v) + C\Psi v && \text{[def. of } v\text{]} \\ &= \Psi(\partial_s v + i\partial_t v) + (\partial_s\Psi + J\partial_t\Psi + C)v \end{aligned}$$

$= \Psi(\partial_s v + i\partial_t v + Cv)$ [def. of C].

The statement about the regularity of C is true because Ψ is of class $W^{1,p}$ and the expression of C has first derivatives of Ψ.

5. There exists $\delta \in (0,\varepsilon)$ and $\tilde{\Phi} \in W^{1,p}(B_\varepsilon, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n,\mathbb{C}^n))$ such that
   - $\forall z \in B_\delta$: $\tilde{\Phi}(z)$ is invertible;
   - $\forall z \in B_\delta$: $i\tilde{\Phi}(z) = \tilde{\Phi}(z)i$;
   - The map $\tilde{\sigma}(z) \coloneqq \tilde{\Phi}(z)^{-1}v(z)$ is holomorphic.

   Proof: By step 4 the data satisfies the hypothesis of the theorem. By assumption the theorem is true for $J = i$.
6. Let $\Phi \coloneqq \Psi\tilde{\Phi}$. Then,
   - $\forall z \in B_\delta$: $\Phi(z)$ is invertible;
   - $\forall z \in B_\delta$: $J(z)\Phi(z) = \Phi(z)i$;
   - The map $\sigma(z) \coloneqq \Phi(z)^{-1}u(z)$ is holomorphic.

   Proof: $\Phi(z)$ is invertible because it is the product of invertible maps. $J(z)\Phi(z) = \Phi(z)i$ because Ψ and $\tilde{\Phi}$ satisfy the same equation. Since
$$\begin{aligned}\sigma(z) &= \Phi(z)^{-1}u(z)\\ &= (\tilde{\Phi}(z))^{-1}\Psi(z)^{-1}u(z)\\ &= \tilde{\Phi}(z)^{-1}v(z)\\ &= \tilde{\sigma}(z)\end{aligned}$$
   and $\tilde{\sigma}$ is holomorphic, then σ is holomorphic.
7. Q.E.D.

   Proof: Steps 1 and 6. □

Exercise 8.2. Show that we may assume that C is i-linear.

Proof:
1. It suffices to prove that there exists a function $A \in L^p(B_\varepsilon, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n,\mathbb{C}^n))$ such that $\forall z \in B_\varepsilon$: $A(z)u(z) = C(z)u(z)$.

   Proof: If such a function exists, then we can apply the theorem in the case that C is i-linear, which we are assuming to be true, to get the result.
2. Let $C^\pm \coloneqq \frac{1}{2}(C \mp iCi)$.
3. For each $D \in L^\infty(B_\varepsilon, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n,\mathbb{C}^n))$, define
$$A_D \coloneqq C^+ + C^-D \in L^p(B_\varepsilon, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n,\mathbb{C}^n)).$$
4. If D is complex anti-linear then A_D is complex linear.

   Proof: By definition of $C^\pm$, $C^\pm i = \pm iC^\pm$.
$$\begin{aligned}&D \text{ is complex anti-linear}\\ &\iff Di = -iD && \text{[def. of complex anti-linear]}\\ &\implies C^-Di = -C^-iD\\ &\iff C^-Di = iC^-D && [C^\pm i = \pm iC^\pm]\\ &\iff C^+i + C^-Di = iC^+ + iC^-D && [C^\pm i = \pm iC^\pm]\end{aligned}$$

$$\begin{aligned} &\iff (C^+ + C^- D)i = i(C^+ + C^- D) \\ &\iff A_D i = iA_D && \text{[def. of } A_D] \\ &\iff A_D \text{ is complex linear.} \end{aligned}$$

5. If $D(z)u(z) = u(z)$, then $A_D(z)u(z) = C(z)u(z)$.

   Proof:
$$\begin{aligned} A_D(z)u(z) &= \Big(C^+(z) + C^-(z)D(z)\Big)u(z) && \text{[def. of } A_D] \\ &= \Big(C^+(z) + C^-(z)\Big)u(z) && \text{[hypothesis]} \\ &= C(z)u(z) && \text{[def. of } C^\pm]. \end{aligned}$$

6. The map
$$D_0(z)\xi = \begin{cases} |u(z)|^{-2}u(z)u(z)^T\bar{\xi} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0, \end{cases}$$
   satisfies:
   - it is complex anti-linear;
   - it is in $L^\infty(B_\varepsilon, \mathrm{Hom}_\mathbb{R}(\mathbb{C}^n, \mathbb{C}^n))$;
   - $D_0(z)u(z) = u(z)$.

   Proof:

   6.1. D_0 is complex anti-linear.

   Proof:
$$\begin{aligned} D_0(z)i\xi &= \begin{cases} |u(z)|^{-2}u(z)u(z)^T\bar{i\xi} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0, \end{cases} \\ &= \begin{cases} -i|u(z)|^{-2}u(z)u(z)^T\bar{\xi} & \text{if } u(z) \neq 0 \\ -i0 & \text{if } u(z) = 0, \end{cases} \\ &= -iD_0(z)\xi. \end{aligned}$$

   6.2. D_0 is of class L^∞.

   Proof: It suffices to show that D_0 is bounded.
$$\begin{aligned} \left\| \frac{u(z)u(z)^T}{|u(z)|^2} \right\| &= \frac{1}{|u(z)|^2}\|u(z)u(z)^T\| \\ &\leq \frac{1}{|u(z)|^2}\|u(z)\|\|u(z)^T\| \\ &= \frac{1}{|u(z)|^2}|u(z)| \max_{\|\xi\|=1}|\langle u(z), \xi\rangle| \\ &= \frac{1}{|u(z)|^2}|u(z)||u(z)| && \text{[Cauchy–Schwarz, equality case]} \\ &= 1. \end{aligned}$$

   6.3. $D_0 u(z) = u(z)$.

   Proof:
$$\begin{aligned} D_0(z)u(z) &= \begin{cases} |u(z)|^{-2}u(z)u(z)^T\bar{u(z)} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0, \end{cases} \\ &= \begin{cases} |u(z)|^{-2}u(z)|u(z)|^2 & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0, \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} u(z) & \text{if } u(z) \neq 0 \\ u(z) & \text{if } u(z) = 0, \end{cases} \\
&= u(z).
\end{aligned}
$$

6.4. Q.E.D.

Proof: Steps 6.1, 6.2 and 6.3.

7. Q.E.D.

Proof: By steps 4, 5 and 6, A_{D_0} is the desired map in step 1. □

Exercise 8.3. Prove the Carleman similarity principle in the case $J(z) = i$ and C is i-linear.

Proof:

1. For $\delta \in (0, \varepsilon)$, define $C_\delta \in L^p(S^2, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n))$ by
$$C_\delta(z) = \begin{cases} C(z) & \text{if } z \in B_\delta \\ 0 & \text{if } z \in S^2 \backslash B_\delta. \end{cases}$$
2. Let $V := \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$. For $\delta \in (0, \varepsilon)$, define
$$\begin{aligned} D_\delta \colon W^{1,p}(S^2, V) &\longrightarrow L^p(S^2, \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, V)) \\ \Phi &\longmapsto \bar{\partial}\Phi + C_\delta \Phi d\bar{z}. \end{aligned}$$
3. For $\delta \in (0, \varepsilon)$, define
$$\begin{aligned} D_\delta^{\mathrm{ev}} \colon W^{1,p}(S^2, V) &\longrightarrow L^p(S^2, \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, V)) \times V \\ \Phi &\longmapsto (\bar{\partial}\Phi + C_\delta \Phi d\bar{z}, \Phi(0)). \end{aligned}$$
4. D_0^{ev} is bijective.

   Proof: Riemann-Roch theorem.
5. There exists $\delta' \in (0, \varepsilon)$ such that for all $\delta \in (0, \delta')$, D_δ^{ev} is bijective.

   Proof:

   5.1. There exists an $\varepsilon' > 0$ such that for all
   $$E \colon W^{1,p}(S^2, V) \longrightarrow L^p(S^2, \overline{\mathrm{Hom}}_{\mathbb{C}}(TS^2, V)) \times V$$
   with $\|D_0^{\mathrm{ev}} - E\| < \varepsilon'$, E is bijective.

   Proof: The set of bijective operators between Banach spaces is open in the operator topology.

   5.2. There exists a $\delta' \in (0, \varepsilon)$ such that for all $\delta \in (0, \delta')$, $\|D_0^{\mathrm{ev}} - D_\delta^{\mathrm{ev}}\| < \varepsilon'$.

   Proof: $\lim_{\delta \to 0} \|C_\delta\|_{L^p} = 0$.

   5.3. Q.E.D.

   Proof: By steps 5.1 and 5.2.
6. For each $\delta \in (0, \delta')$ there exists a unique $\Phi_\delta \in W^{1,p}(S^2, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n))$ such that $D_\delta \Phi_\delta = 0$ and $\Phi_\delta(0) = \mathrm{id}_{\mathbb{C}^n}$.

   Proof: By step 5.
7. There exists a $\delta \in (0, \delta')$ such that for all $z \in S^2$, $\Phi_\delta(z)$ is invertible.

   Proof:

   7.1. $\bar{\partial}\Phi_\delta + C_\delta \Phi_\delta d\bar{z} = 0$ in B_δ and $\bar{\partial}\Phi_\delta = 0$ in $S^2 \backslash B_\delta$.

Proof: Since $D_\delta \Phi_\delta = 0$.

7.2. $\lim_{\delta\to 0} \|\Phi_\delta - \mathrm{id}_{\mathbb{C}^n}\|_{W^{1,p}} = 0$.

Proof: By step 7.1, as $\delta \to 0$ Φ_δ converges to a solution of $\bar\partial \Phi = 0$, $\Phi(0) = \mathrm{id}_{\mathbb{C}^n}$, which is $\Phi(z) = \mathrm{id}_{\mathbb{C}^n}$ by step 3.

7.3. Q.E.D.

Proof: By step 7.2, it suffices to choose Φ_δ so near $\mathrm{id}_{\mathbb{C}^n}$ in $W^{1,p}$ that

$$\|\det \Phi - 1\|_{C^0} < 1/2.$$

8. Let $\Phi := \Phi_\delta$. Then Φ is as desired, i.e.
   (i) *(i)* $\forall z \in B_\delta\colon \Phi(z)$ is invertible;
   (ii) *(ii)* $\forall z \in B_\delta\colon J(z)\Phi(z) = \Phi(z)i$;
   (iii) *(iii)* The map

$$\begin{aligned} \sigma\colon B_\delta &\longrightarrow \mathbb{C}^n \\ z &\longmapsto \Phi(z)^{-1}u(z) \end{aligned}$$

is holomorphic.

Proof:

8.1. $\forall z \in B_\delta\colon \Phi(z)$ is invertible.

Proof: By step 7.

8.2. $\forall z \in B_\delta\colon i\Phi(z) = \Phi(z)i$.

Proof: By the construction of Φ in step 6, $\Phi \in W^{1,p}(S^2, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n))$.

8.3. σ is holomorphic.

Proof:

$$\begin{aligned} 0 &= \partial_s u + i\partial_t u + Cu && \text{[by hypothesis]} \\ &= \partial_s(\Phi\sigma) + i\partial_t(\Phi\sigma) + C\Phi\sigma && \text{[by definition of } \sigma\text{]} \\ &= \Phi(\partial_s\sigma + i\partial_t\sigma) + (\partial_s\Phi + i\partial_t\Phi + C)\sigma && \text{[Leibnitz rule]} \\ &= \Phi(\partial_s\sigma + i\partial_t\sigma) && \text{[By step 6 and } C = C_\delta \text{ on } B_\delta\text{].} \end{aligned}$$

8.4. Q.E.D.

Proof: Steps 8.1, 8.2 and 8.3.

9. Q.E.D.

Proof: By step 8. $\square$

9 Exercise sheet No. 9 - 27-06-2019

Exercise 9.1. Let Σ, Σ' be Riemann surfaces and $\varphi\colon \Sigma \longrightarrow \Sigma'$ be a holomorphic map. Show that for all $p \in \Sigma$ there exists a unique $n \in \mathbb{N}_0$ such that there exist coordinate neighbourhoods of p in Σ and of $\varphi(p)$ in Σ' such that with respect to these coordinates ϕ is given by $\phi(z) = z^n$.

Proof sketch: Start by writing φ locally as it's Taylor series: $\varphi(z) = \sum_{j=0}^{\infty} a_j z^j$. Let $n = \min\{j \mid a_j \neq 0\}$ and $t(z) = \sum_{j=n}^{\infty} a_j z^{j-n}$ (so $t(z)$ is such that $\varphi(z) = t(z)z^n$). Find some r holomorphic such that $r^n = t$ and let $\alpha(z) = zr$. By our choice of n, α is a local biholomorphism: $\alpha'(0)^k = r(0)^k = t(0) = a_n \neq 0$. Then $\varphi(z) = z^n t(z) = z^n r^n(z) = (\alpha(z))^n$, and composing the coordinate chart that we had with α gives the desired coordinates. See figure 1.

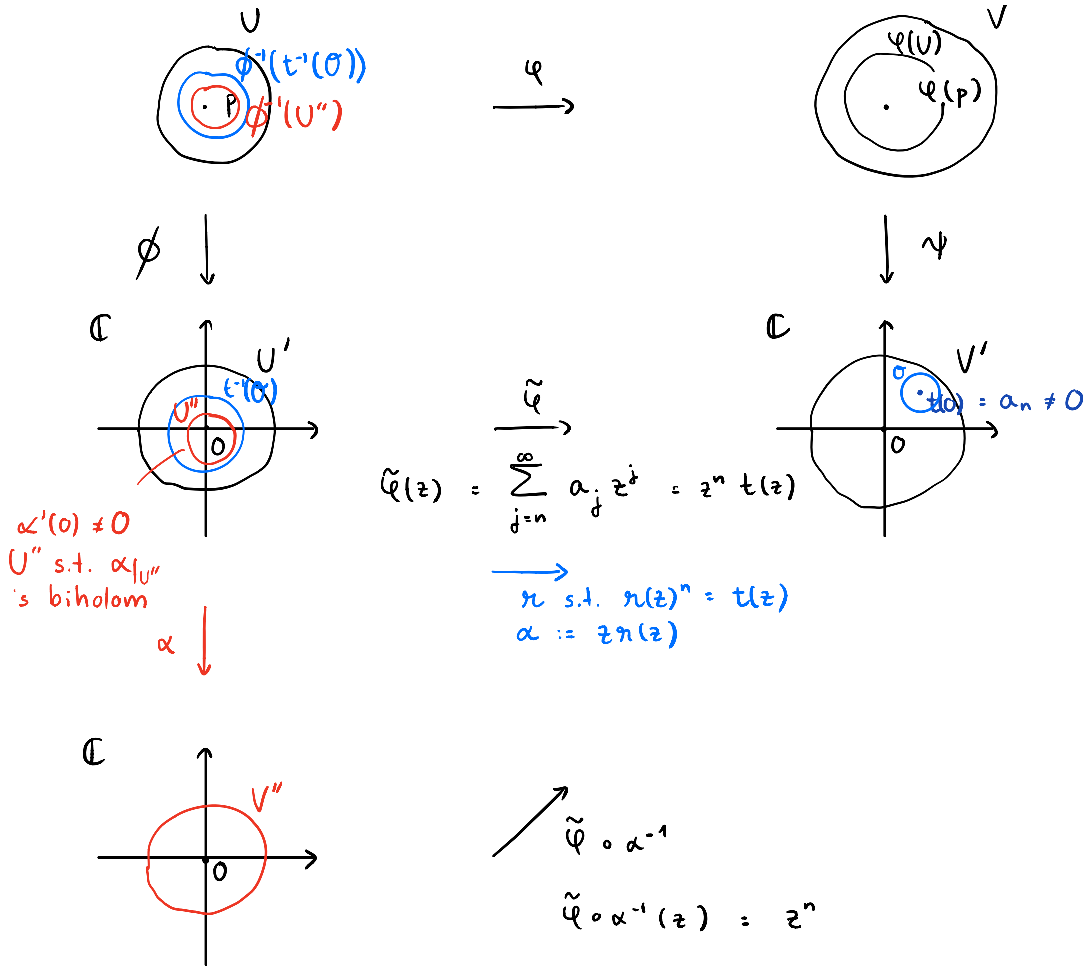


Figure 1: Proof sketch of existence.

For uniqueness, note that n is the number of points in the preimage of a carefully chosen point, see figure 2.

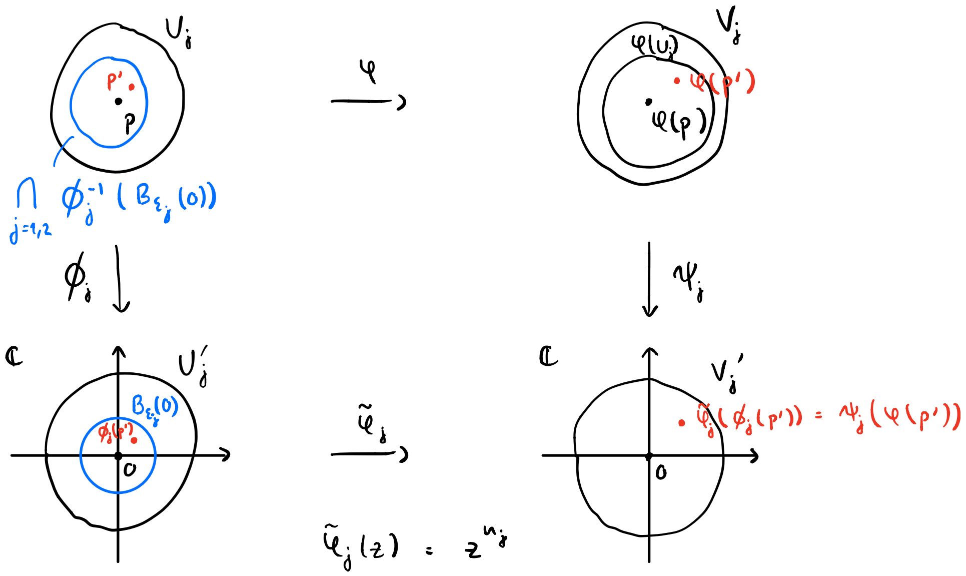


Figure 2: Proof sketch of uniqueness.

Solution:

1. There exist coordinate charts $(U \subset \Sigma, U' \subset \mathbb{C}, \phi\colon U \longrightarrow U')$ around p and $(V \subset \Sigma, V' \subset \mathbb{C}, \psi\colon V \longrightarrow V')$ around $\varphi(p)$ such that $\tilde{\varphi} := \psi \circ \varphi \circ \phi^{-1}\colon U' \longrightarrow V'$ is given on U' by its Taylor series expansion.

   Proof: Take any coordinate charts around p, $\varphi(p)$, and restrict the domain to be contained inside the radius of convergence of the holomorphic function $\tilde{\varphi}$.

2. We may assume that $\tilde{\varphi}$ is not constant.

   Proof: If $\tilde{\varphi}$ is constant then $n = 0$ is as desired. If $n' \in \mathbb{N}_0$ is other number with the same property, then it must be 0 because locally $\varphi(z) = z^{n'}$ is constant.

3. Let
   - $\{a_j\}_{j\in\mathbb{N}}$ be the coefficients of the power series of $\tilde{\varphi}$;
   - $n = \min\{j \mid a_j \neq 0\}$;
   - t be given by

   $$\begin{aligned} t\colon U' &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{j=n}^{\infty} a_j z^{j-n}. \end{aligned}$$

4. We may assume that here exists a holomorphic function $r\colon U' \longrightarrow \mathbb{C}$ such that $r(z)^n = t(z)$.

   Proof: Let O be an open ball around $a_n = t(0) \neq 0$ contained in V such that $0 \neq O$, and restrict U' to $t^{-1}(O)$ and U such that ϕ is still bijective. On $t^{-1}(O)$, we can define an n-th root of t because $t(t^{-1}(O)) = O$ is a small enough ball that does not intersect 0.

5. Let $\alpha\colon U' \longrightarrow \mathbb{C}$ be given by $\alpha = zr(z)$.

6. There exists $U'' \subset U$ a neighborhood of 0, $V'' \subset \mathbb{C}$ a neighborhood of 0, such that $\alpha|_{U''}\colon U'' \longrightarrow V''$ is a biholomorphism.

Proof: α is holomorphic,

$$\begin{aligned} \alpha'(0) &= r(0) && [\text{by def. of } \alpha] \\ &\neq 0 && [\text{by } t(0) = a_n \neq 0], \end{aligned}$$

and the inverse function theorem.

7. We may assume w.l.o.g. that $U'' = U'$.

   Proof: Shrink U' to $U'' \ni 0$ and U such that ϕ is still bijective.

8. Existence of n.

   Proof: In the coordinate neighborhoods

   $$(U \subset \Sigma, U'' \subset \mathbb{C}, \alpha \circ \phi \colon U \longrightarrow V'') \text{ of } p,$$
   $$(V \subset \Sigma, V' \subset \mathbb{C}, \psi \colon V' \longrightarrow V) \text{ of } \varphi(p),$$

   $$\begin{aligned} \psi \circ \varphi \circ (\alpha \circ \phi)^{-1}(z) &= \psi \circ \varphi \circ \phi^{-1} \circ \alpha^{-1}(z) \\ &= (\alpha(\alpha^{-1}(z)))^n \\ &= z^n. \end{aligned}$$

9. Uniqueness of n.

   Proof: It suffices to assume that for $j = 1, 2$,

   $$(U_j \subset \Sigma, U_j' \subset \mathbb{C}, \phi_j \colon U_j \longrightarrow U_j') \text{ is a coordinate neighborhood of } p,$$
   $$(V_j \subset \Sigma, V_j' \subset \mathbb{C}, \psi \colon V_j \longrightarrow V_j') \text{ is a coordinate neighborhood of } \psi(p),$$

   such that with respect to these coordinate neighboorhoods $\varphi(z) = z^{n_j}$, and prove that $n_1 = n_2$. For $j = 1, 2$ let $\varepsilon_j > 0$ be such that $B_{\varepsilon_j}(0) \subset U_j'$. Let $p' \in \bigcap_{j=1}^2 \phi_j^{-1}(B_{\varepsilon_j}(0) \backslash \{0\})$. Then

   $$\begin{aligned} \#\varphi|_U^{-1}(\varphi(p)) &= \#\tilde{\varphi}_j^{-1}(\psi_j(\varphi(p'))) \\ &= \#\{z \in U_j' \mid z^{n_j} = \psi_j(\varphi(p'))\} \\ &= n_j, \end{aligned}$$

   where in the last equality we used the fact that there are n n-th roots of a complex number and they are in a circle centered at 0, $B_{\varepsilon_j}(0) \subset U_j'$ and $\phi_j(p') \subset B_{\varepsilon_j}(0) \backslash \{0\}$. So, $n_1 = \#\varphi|_U^{-1}(\varphi(p)) = n_2$.

10. Q.E.D.

    Proof: Steps 8 and 9. □

Exercise 9.1 allows us to define the **multiplicity** of φ at $p \in \Sigma$, $\operatorname{mult}_p \varphi$, by

$$\operatorname{mult}_p \varphi = \begin{cases} \text{the unique } n \in \mathbb{N}_0 \text{ as in exercise 9.1} & \text{if that n is not 0} \\ \infty & \text{if that n is 0.} \end{cases}$$

Exercise 9.2. Let Σ, Σ' be connected Riemann surfaces and $\varphi \colon \Sigma \longrightarrow \Sigma'$ be a holomorphic map. Show that if there exists $p \in \Sigma$ such that $\operatorname{mult}_p \varphi = \infty$, then for all $p \in \Sigma$ $\operatorname{mult}_p \varphi = \infty$.

Proof sketch: Let $S := \{p \in \Sigma \mid \operatorname{mult}_p \varphi = \infty\}$. By exercise 9.1, S is open and closed. By hypothesis it is nonempty. Since Σ is connected, $S = \Sigma$.

Solution:

1. Let $S \coloneqq \{p \in \Sigma \mid \operatorname{mult}_p \varphi = \infty\}$.
2. S is open.

   Proof: It suffices to assume that $p \in S$ and prove that there exists a neighborhood of p contained in S. By exercise 9.1 and the fact that $\operatorname{mult}_p \varphi = \infty$, there exist coordinate neighborhoods U centered at p and V centered at $\varphi(p)$ such that with respect to these coordinate neighborhoods $\varphi(z) = 0$. Then, $\forall q \in U\colon \operatorname{mult}_q \varphi = \infty$. So U is the desired open neighborhood.
3. S is closed.

   Proof: We show that $\Sigma \backslash S$ is open. If suffices to assume that $p \in \Sigma\backslash S$ and prove that there exists a neighborhood of p contained in $\Sigma\backslash S$. Let $n = \operatorname{mult}_p \varphi < \infty$. By exercise 9.1 and the fact that $n = \operatorname{mult}_p \varphi < \infty$, there exist coordinate neighborhoods U centered at p and V centered at $\varphi(p)$ such that with respect to these coordinate neighborhoods $\varphi(z) = z^n$. Then, $\forall q \in U\colon \operatorname{mult}_q \varphi < \infty$, because if it were ∞ for some q then on a neighborhood of q φ would be constant, contradicting $\varphi(z) = z^n$. So U is the desired open neighborhood.
4. Q.E.D.

   Proof:
   $$\begin{aligned}
   &\exists p \in \Sigma\colon \operatorname{mult}_p \varphi = \infty \\
   &\quad\iff S \neq \varnothing \\
   &\quad\iff S = \Sigma && [\Sigma \text{ is connected, } S \neq \varnothing,\ S \text{ is open and closed}] \\
   &\quad\iff \forall p \in \Sigma\colon \operatorname{mult}_p \varphi = \infty.
   \end{aligned}$$
   □

Exercise 9.3. Let Σ, Σ' be compact, connected Riemann surfaces, and $\varphi\colon \Sigma \longrightarrow \Sigma'$ be a nonconstant holomorphic map. Show that for all $q \in \Sigma'$

$$\deg(\varphi) = \sum_{p \in \varphi^{-1}(q)} \operatorname{mult}_p \varphi.$$

Proof sketch: For each $q \in \Sigma'$, by exercise 9.1, $\varphi^{-1}(q)$ is discrete, therefore finite by compactness of Σ. Therefore the sum on the right hand side is in $\mathbb{N}$. To prove the result it suffices to show that this sum does not depend on the chosen q, because if q is a regular value then $\forall p \in \varphi^{-1}(q)\colon \operatorname{mult}_p \varphi = 1$ and the sum is equal to the degree of φ. To prove that the sum is constant, it suffices to show that it is locally constant, by connectedness of Σ'. Let then $q \in \Sigma'$, $\varphi^{-1}(q) = \{p_1, \dots, p_k\}$. Then, by exercise 9.1, there exist $U_1, \dots, U_k$ pairwise disjoint coordinate neighborhoods of $p_1, \dots, p_k$, V a coordinate neighborhood of q such that

(i) with respect to coordinates centered at p_j, q, $\varphi(z) = z^{d_j}$.

(ii) $\varphi|_{U_j\backslash\{p_j\}}\colon U_j\backslash\{p_j\} \longrightarrow \varphi(U_j)\backslash\{q\}$ is a d_j-fold covering map.

Using (i) and (ii) we can show that for all $q' \in V$,
$$\deg'_\varphi(q') = d_1 + \cdots + d_k,$$
both in the case where $q' = q$ and in the case where $q' \neq q$.
Solution:

1. For all $q \in \Sigma'$, $\varphi^{-1}(q)$ is finite.

   Proof:

   1.1. $\varphi^{-1}(q)$ is discrete.

   Proof: It suffices to assume that $p \in \varphi^{-1}(q)$ and prove that there exists a neighborhood U of p such that $U \cap \varphi^{-1}(q) = \{p\}$. By exercise 9.1, there exist coordinate neighborhoods U centered at p and V centered at $\varphi(q)$ such that with respect to these coordinates φ is given by $\varphi(z) = z^n$. Since φ is nonconstant, $n > 0$. Then, $U \cap \varphi^{-1}(q) = \{p\}$, because $z^n = 0 \iff z = 0$.

   1.2. $\varphi^{-1}(q)$ is compact.

   Proof: q is closed, therefore $\varphi^{-1}(q)$ is closed, and $\varphi^{-1}(q) \subset \Sigma$ which is compact.

   1.3. Let X be a topological space and K be a compact, discrete subset. Then K is finite.

   Proof: Assume by contradiction that K is infinite. For each $x \in K$, let U_x be a neighborhood of x such that $U_x \cap K = \{x\}$. Then $\{U_x\}_{x \in K}$ is an open cover of K that has no subcovers. Contradiction.

   1.4. Q.E.D.

   Proof: Steps 1.1, 1.2 1.3.

2. The map $\deg'_\varphi(q) \coloneqq \sum_{p \in \varphi^{-1}(q)} \operatorname{mult}_p \varphi$ is constant.

   Proof:

   2.1. It suffices to show that $\deg'_\varphi$ is locally constant, i.e. that for each $q \in \Sigma'$ there exists a neighborhood U_q of q such that $\deg'_\varphi|_{U_q}$ is constant.

   Proof: It suffices to assume that $q_0, q_1 \in \Sigma'$ and prove that $\deg'_\varphi(q_0) = \deg'_\varphi(q_1)$. Since Σ' is locally path connected (it is a manifold) and it is connected, it is path connected. Let γ be a path joining q_0 and q_1. For each q in the image of γ, let U_q be a neighborhood of q such that $\deg'_\varphi|_{U_q}$ is constant. Then $\{U_q\}_{q \in \operatorname{im}\gamma}$ is an open cover of $\operatorname{im}\gamma$. Let $U_{q_0}, U_{q_1}, U_{q_1^*}, \ldots, U_{q_N^*}$ be a finite subcover. Then $\deg'_\varphi(q_0) = \deg'_\varphi(q_1)$.

   2.2. Let $\{p_1, \ldots, p_k\} \coloneqq \varphi^{-1}(q)$.

   Proof: By step 1, $\varphi^{-1}(q)$ is finite.

   2.3. For each $j = 1, \ldots, k$, let U_j be a coordinate neighborhood centered at p_j whose corresponding neighborhood in $\mathbb{C}$ is a ball, and V_j be a coordinate neighborhood centered at $\varphi(p_j) = q$ such that

   - with respect to these coordinates, $\varphi(z) = z^{d_j}$;
   - $\forall j \colon \varphi^{-1}(V_j) \subset \bigcup_{l=1}^k U_l$.

   Proof: By exercise 9.1 applied to each p_j.

   2.4. $\forall j = 1, \ldots, k \colon \exists V'_j \subset V_j$ an open neighborhood of q such that
   $$\varphi|_{U_j \setminus \{p_j\}} \colon U_j \setminus \{p_j\} \longrightarrow V'_j \setminus \{q\}$$
   is a d_j-fold covering map.

   Proof sketch: See figure 3.

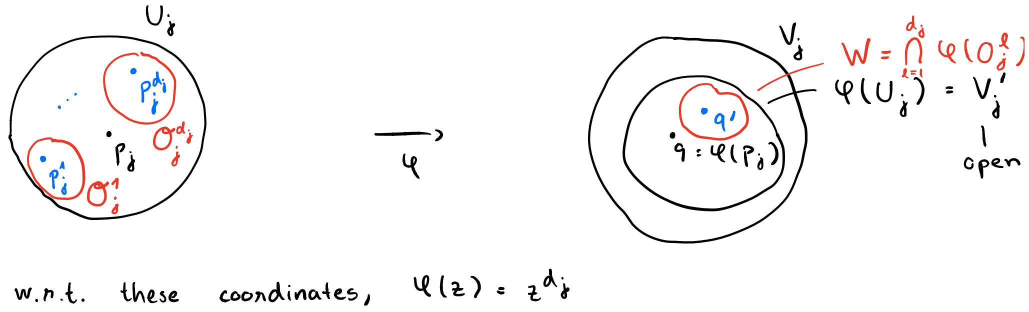


Figure 3: Proof sketch of covering map.

Proof: Let $V_j' = \varphi(U_j) \subset V_j$. $\varphi(p_j) = q$, $\varphi^{-1}(q) \cap U_j = \{p_j\}$. By the coordinate representation $\varphi(z) = z^{d_j}$ and the fact that the image of U_j in $\mathbb{C}$ is a ball, the image of $\varphi(U_j)$ in $\mathbb{C}$ is a ball as well and V_j' is open. Then,

$$\varphi|_{U_j \setminus \{p_j\}} \colon U_j \setminus \{p_j\} \longrightarrow V_j' \setminus \{q\}$$

is well defined and surjective. Let $q' \in V_j' \setminus \{q\}$. We must show that q' has a neighborhood that is evenly covered by φ. Since $\varphi(z) = z^{d_j}$ in $U_j \setminus \{p_j\}$ and U_j is a ball, $\varphi|_{U_j \setminus \{p_j\}}^{-1}(q')$ is a set consisting of d_j points $p_j^1, \dots, p_j^{d_j}$. On each of the $p_j^1, \dots, p_j^{d_j}$ the derivative of φ is nonzero, so there exist open sets $O_j^l \ni p_j^l$ for $l = 1, \dots, d_j$ such that

$$\varphi|_{O_j^l} \colon O_j^l \longrightarrow \varphi(O_j^l)$$

is a diffeomorphism. Let $W = \bigcap_{l=1}^{d_j} \varphi(O_j^l)$. Then W is a neighborhood of q that is evenly covered by φ.

2.5. Let $V := \bigcap_{j=1}^{n} V_j'$. Then $\forall q \in V\colon\ \deg_\varphi'(q') = d_1 + \dots + d_k$.

Proof:

2.5.1. Case $q' = q$.

Proof:

$$\begin{aligned}
\deg_\varphi'(q) &= \sum_{p \in \varphi^{-1}(q)} \operatorname{mult}_p(\varphi) \\
&= \operatorname{mult}_{p_1} \varphi + \dots + \operatorname{mult}_{p_k} \varphi \\
&= d_1 + \dots + d_k.
\end{aligned}$$

2.5.2. Case $q' \neq q$.

Proof:

$$\begin{aligned}
\deg_\varphi'(q') &= \sum_{p \in \varphi^{-1}(q')} \operatorname{mult}_p \varphi \\
&= \sum_{j=1}^{k} \sum_{l=1}^{d_j} \operatorname{mult}_{p_j^l} \varphi \\
&= \sum_{j=1}^{k} \sum_{l=1}^{d_j} 1 \\
&= d_1 + \dots + d_k.
\end{aligned}$$

2.5.3. Q.E.D.

Proof: Steps 2.5.1 and 2.5.2

3. $\forall q \in \Sigma'\colon\ \deg(\varphi) = \deg'_\varphi(q)$.

   Proof: Let q be given. Let q' be a regular value of φ.

$$\begin{aligned}
\deg'_\varphi(q) &= \deg'_\varphi(q') && \text{[step 2]} \\
&= \sum_{p \in \varphi^{-1}(q')} \operatorname{mult}_p \varphi && \text{[definition of } \deg'_\varphi\text{]} \\
&= \sum_{p \in \varphi^{-1}(q')} 1 && [q' \text{ is a regular value}] \\
&= \#\varphi^{-1}(q') \\
&= \deg \varphi && [\text{definition of } \deg + \varphi \text{ is holomorphic}].
\end{aligned}$$

4. Q.E.D.

   Proof: Steps 2 and 3. $\square$

10 Exercise sheet No. 10 - 09-07-2019

In this exercise sheet we are going to study the concept of simple and multiply covered curves. We will use the results from exercise sheet No. 9. We follow the presentation given in [Wen15].

Exercise 10.1. Let Σ, Σ' be closed connected Riemann surfaces. Let $\varphi\colon \Sigma \longrightarrow \Sigma'$ be holomorphic. Show that

(i) $\deg\varphi \geq 0$;

(ii) $\deg\varphi = 0$ if and only if φ is constant;

(iii) $\deg\varphi = 1$ if and only if φ is a biholomorphism;

(iv) $\deg\varphi = k \geq 2$ if and only if

  (a) $\mathrm{CritPts}(\varphi)$ is finite;

  (b) $\varphi|_{\varphi^{-1}(\Sigma'\setminus\mathrm{CritVal}(\varphi))}\colon \varphi^{-1}(\Sigma'\setminus\mathrm{CritVal}(\varphi)) \longrightarrow \Sigma'\setminus\mathrm{CritVal}(\varphi)$ is a k-fold covering map;

  (c) $\forall z \in \mathrm{CritPts}(\varphi)$:
  $\exists! l \in \{2,\dots,k\}$:
  $\exists U$ a coordinate neighborhood centered at z:
  $\exists U'$ a coordinate neighborhood centered at $\varphi(z)$:
  with respect to these coordinates φ is given by $\varphi(z) = z^l$.

Proof sketch: Use the description of the degree from exercise 9.3

$$\deg\varphi = \sum_{p\in\varphi^{-1}(q)} \mathrm{mult}_p\,\varphi$$

and the coordinates from exercise 9.1/the definition of $\mathrm{mult}_p\,\varphi$.
Solution:
1. (i):

   Proof: Let $q' \in \Sigma'$.

$$\begin{aligned}\deg\varphi &= \sum_{p\in\varphi^{-1}(q)} \mathrm{mult}_p\,\varphi && \text{[exercise 9.3]}\\ &\geq 0 && \text{[By def. of mult]}.\end{aligned}$$

2. (ii):

   Proof:
   2.1. ($\Longleftarrow$):

   Proof: Let y be the only point in the image of φ, and $y' \in \Sigma\setminus\{y\}$. Then, y' is regular.

$$\begin{aligned}\deg\varphi &= \#\varphi^{-1}(y')\\ &= 0.\end{aligned}$$

   2.2. ($\Longrightarrow$):

Proof: Assume by contradiction that φ is nonconstant. Then, since $\deg\varphi = 0$,
$$\forall y \in \Sigma' \colon \sum_{x\in\varphi^{-1}(y)} \operatorname{mult}_x \varphi = 0.$$
Since φ is nonconstant, by exercise 9.2 $\operatorname{mult}_x \varphi \in \mathbb{N}$, therefore
$$\forall y \in \Sigma' \colon \varphi^{-1}(y) = \varnothing.$$
Contradiction.

2.3. Q.E.D.

Proof: Steps 2.1 and 2.2.

3. (iii):

Proof:
$$\begin{aligned}
\deg\varphi = 1 &\iff \forall y \in \Sigma' \colon \sum_{x\in\varphi^{-1}(y)} \operatorname{mult}_x \varphi = 1 && [9.3]\\
&\iff \forall y \in \Sigma' \colon \exists! x \in \varphi^{-1}(y) \text{ and } \operatorname{mult}_x \varphi = 1 && [\text{by } 9.2,\ \operatorname{mult}_x \varphi \geq 1]\\
&\iff \varphi \text{ is bijective and } \forall x \in \Sigma \colon \operatorname{mult}_x \varphi = 1\\
&\iff \varphi \text{ is bijective and a local biholomorphism}\\
&\iff \varphi \text{ is a biholomorphism.}
\end{aligned}$$

4. (iv):

Proof:

4.1. ($\Longleftarrow$):

Proof: Let $y \in \Sigma'$ be a regular value. Then, since φ is a k-fold branched covering, there exists V a neighborhood of y in $\Sigma' \setminus \operatorname{CritVal}(\varphi)$ and there exist $U_1, \ldots, U_k$ pairwise disjoint open in $\Sigma' \setminus \operatorname{CritPts}(\varphi)$ such that

(i) $\varphi^{-1}(V) = \bigcup_{j=1}^{k} U_j$,

(ii) $\varphi|_{U_j} \colon U_j \longrightarrow V$ is a homeomorphism.

Then,
$$\begin{aligned}
\deg\varphi &= \#\varphi^{-1}(y) && [\text{def. of deg} + \varphi \text{ is holomorphic}]\\
&= k. && [\text{(i) and (ii)}].
\end{aligned}$$

4.2. ($\Longrightarrow$):

Proof:

4.2.1. (c):

Proof:

4.2.1.1. It suffices to show that $\operatorname{mult}_z \varphi \neq \infty$ and $\operatorname{mult}_z \varphi \neq 1$.

Proof: By exercise 9.3, $l = \operatorname{mult}_z \varphi \in \{2, \ldots, k\}$. The result follows from exercise 9.1.

4.2.1.2. $\operatorname{mult}_z \varphi \neq \infty$.

Proof: Assume by contradiction that $\operatorname{mult}_z \varphi = \infty$. Then φ is constant, by exercise 9.2. By 2, $\deg\varphi = 0$. Contradiction.

4.2.1.3. $\operatorname{mult}_z \varphi \neq 1$.

Proof: Assume by contradiction that $\operatorname{mult}_z \varphi = 1$. Then $d\varphi(z)$ is an isomorphism. But $d\varphi(z) = 0$ because z is a critical point.

Contradiction.

4.2.1.4. Q.E.D.

Proof: Steps 4.2.1.1, 4.2.1.2 and 4.2.1.3.

4.2.2. (a):

Proof: Since Σ is compact, it suffices to show that $\operatorname{CritPts}(\varphi)$ is discrete. For this, it suffices to assume that $p \in \operatorname{CritPts}(\varphi)$ and prove that there exists a neighborhood U of p such that $U \cap \operatorname{CritPts}(\varphi) = \{p\}$. By exercise 9.1, there exist coordinate neighborhoods U centered at p and V centered at $\varphi(p)$ such that with respect to these coordinates $\varphi(z) = z^n$. Then U is the desired neighborhood.

4.2.3. (b):

Proof:

4.2.3.1. It suffices to assume that $y \in \Sigma' \setminus \operatorname{CritVal}(\varphi)$ and prove that there exist V a neighborhood of y, $U_1, \ldots, U_k$ pairwise disjoint open sets, such that

(i) $\varphi^{-1}(V) = \bigcup_{j=1}^{k} U_j$,

(ii) $\varphi|_{U_j} : U_j \longrightarrow V$ is a homeomorphism.

Proof: By definition of covering map.

4.2.3.2. There exist $x_1, \ldots, x_k \in \Sigma$ pairwise distinct such that $\varphi^{-1}(y) = \{x_1, \ldots, x_k\}$.

Proof: Since y is a regular value and by definition of deg.

4.2.3.3. For each $j = 1, \ldots, k$, there exists U_j' a coordinate neighborhood centered at x_j and V_j a coordinate neighborhood centered at y such that with respect to these coordinates $\varphi(z) = z$.

Proof: By exercise 9.1, we only have to prove that $\operatorname{mult}_{x_j} \varphi = 1$.

$$\begin{aligned} k &= \deg \varphi && \text{[assumption]} \\ &= \sum_{j=1}^{k} \operatorname{mult}_{x_j} \varphi && \text{[exercise 9.1 and step 4.2.3.3]} \\ \Longrightarrow \forall j \colon 1 &= \operatorname{mult}_{x_j} \varphi && [\operatorname{mult}_{x_j} \varphi \geq 1]. \end{aligned}$$

4.2.3.4. We may assume that $U_1', \ldots, U_k'$ are pairwise disjoint.

Proof: Since the $x_1, \ldots, x_k$ are pairwise distinct and Σ is Hausdorff.

4.2.3.5. We may assume that for each j, $\varphi|_{U_j'} \colon U_j' \longrightarrow V_j$ is a biholomorphism.

Proof: By restricting V_j to $\varphi(U_j')$.

4.2.3.6. Let $V = \bigcap_{j=1}^{k} V_j$ and $U_j = \varphi|_{U_j'}^{-1}(V)$. Then V and $U_1, \ldots, U_k$ are the desired open sets in step 4.2.3.1.

Proof: By steps 4.2.3.4, 4.2.3.5 and the definition of V and U_j.

4.2.3.7. Q.E.D.

Proof: Steps 4.2.3.1 and 4.2.3.6.

4.2.4. Q.E.D.

Proof: Steps 4.2.1, 4.2.2 and 4.2.3.

4.3. Q.E.D.

Proof: Steps 4.1 and 4.2.

5. Q.E.D.

Proof: Steps 1, 2, 3 and 4. □

Exercise 10.2. Let J be a smooth almost complex structure on $\mathbb{C}^n$. Let $u\colon B_1(0) \longrightarrow \mathbb{C}^n$ be a nonconstant smooth J-holomorphic curve with $u(0) = 0$. Then, there exist

- $\varepsilon > 0$,
- $\varphi\colon B_\varepsilon(0) \longrightarrow B_1(0)$ holomorphic such that $\varphi(0) = 0$,
- $v\colon B_1(0) \longrightarrow \mathbb{C}^n$ an injective J-holomorphic curve

such that $u|_{B_\varepsilon(0)} = v \circ \varphi$. Let $j \coloneqq (\phi^{-1})^* i$.

Proof sketch: By the Micallef-White theorem, we can find coordinates ϕ, ψ (see the commutative diagram (1)) and write u locally as a polynomial: if $\hat{u}$ is the local representative of u, $\hat{u}(z) = (z^q, z^q p(z))$. Intuitively speaking, our goal is to compose $\hat{u}$ with a k-th root function to make it injective. We start by stating a fact that describes how much "not injective" $\hat{u}$ is. Let $\psi'_l(z) = ze^{2\pi i l/q}$. Then,

$$\hat{u}(z) = \hat{u}(w) \Longleftrightarrow \exists l = 0, \dots, q-1\colon w = \psi'_l(z) \text{ and } \hat{u} = \hat{u} \circ \psi'_l.$$

Given this fact, define $S = \{l \in \{1, \dots, q\} \mid \hat{u} \circ \psi'_l = \hat{u}\}$ and $m = \min S$. Then, $S = \{m, 2m, \dots, km\}$ where $km = q$. Because of this, we can define $\hat{v}(z) = \hat{u}(z^{1/k})$, which will be injective and continuous. Now naively speaking we would like to define $v = \psi^{-1} \circ \hat{v}$ and $\varphi = \phi$. The problem is that the map ϕ coming from the Micallef-White theorem is not holomorphic. So we must compose it with another map and define v differently. Let then $\Phi\colon (B_\delta(0), j) \longrightarrow (B_1(0), i)$ be a biholomorphism such that $\Phi(0) = 0$. Then Φ is such that $\Phi(ze^{2\pi i l/k}) = \Phi(z) e^{2\pi i l/k}$ (this can be proven examining the group of automorphisms of $B_1(0)$ that map 0 to 0). Let $\tilde{u} = \hat{u} \circ \Phi^{-1}$, and $\tilde{v}(z) = \tilde{u}(z^{1/k})$. Now show that $\tilde{v}$ is injective, smooth and J-holomorphic. For smooth and J-holomorphic note that this is true everywhere except at 0, and for 0 show that $\tilde{v}$ is of class $W^{1,p}$ and use elliptic regularity.

$$
\begin{tikzcd}
\mathbb{C} & \mathbb{C}^n \\
B_1(0) \arrow[u, hook] \arrow[r, "u"] & W \arrow[u, hook] \\
(U, i) \arrow[u, hook] \arrow[d, "\phi"'] & (u(U), i) \arrow[u, hook] \arrow[d, "\psi"] \\
(B_\delta(0), j) \arrow[r, "\hat{u}"] & (\psi(W), J) \\
(B_1(0), i) \arrow[u, "\Phi"] \arrow[ur, "\tilde{u}"] \arrow[d, "(\cdot)^k"'] & \\
(B_1(0), i) \arrow[uur, bend right, "\tilde{v}"'] &
\end{tikzcd}
\tag{1}
$$

Solution:

1. There exist
   - $U \subset B_1(0)$ a neighborhood of 0, $\delta > 0$, $\phi \colon U \longrightarrow B_\delta(0)$ a C^2-diffeomorphism,
   - W a neighborhood of 0 such that $u(U) \subset W$ and $\psi \colon W \longrightarrow \psi(W)$ a C^1-diffeomorphism

   such that $\hat{u} \coloneqq \psi \circ u \circ \phi^{-1} \colon B_\delta(0) \longrightarrow \psi(W)$ is given by
   $$\hat{u}(z) = (z^q, z^q p(z)),$$
   where p is a polynomial such that $p(0) = 0$.

   Proof: By the Micallef-White theorem.

2. Let
   - for each $l \in \mathbb{Z}$,
   $$\begin{aligned} \psi'_l \colon B_\delta(0) &\longrightarrow B_\delta(0) \\ z &\longmapsto z e^{2\pi i l/q}, \end{aligned}$$
   - for each $l \in \mathbb{Z}$, $u'_l = \hat{u} \circ \psi'_l$,
   - $S = \{l \in \{1, \ldots, q\} \mid u'_l = \hat{u}\}$,
   - $m = \min S$.

3. We may assume after restricting δ that for all $z, w \in B_\delta(0)$, $\hat{u}(z) = \hat{u}(w)$ if and only if there exists some $l = 0, \ldots, q-1$ such that $w = z e^{2\pi i l/q}$ and $\hat{u} = u'_l$.

   Proof:

   3.1. We may assume after restricting δ that for all l,
   $$\exists z \in B_\delta(0) \colon \hat{u}(z) = u'_l(z) \iff \forall z \in B_\delta(0) \colon \hat{u}(z) = u'_l(z).$$

   Proof:

3.1.1. $u'_l(z) = (z^q, z^q p'_l(z))$, where $p'_l(z) = p(ze^{2\pi i l/q})$ is a polynomial.

3.1.2. $(\hat{u} - u'_l)(z) = (0, z^q(p(z) - p'_l(z)))$.

3.1.3. It suffices to show that after restricting δ,

$$\begin{aligned} &\exists z \in B_\delta(0)\backslash\{0\}\colon (p - p'_l)(z) = 0 \\ \iff &\forall z \in B_\delta(0)\backslash\{0\}\colon (p - p'_l)(z) = 0. \end{aligned} \tag{2}$$

3.1.4. For each l let δ_l be such that either the only root of $p - p'_l$ in $B_{\delta_l}(0)$ is 0, or $p - p'_l$ is constant in $B_{\delta_l}(0)$. Let $\delta = \min\{\delta_1, \ldots, \delta_{q-1}\}$.

3.1.5. Case: $p - p'_l$ is constant in $B_{\delta_l}(0)$

Proof: Equation (2) is true, because both sides of the equation are true.

3.1.6. Case: the only root of $p - p'_l$ in $B_{\delta_l}(0)$ is 0.

Proof: Equation (2) is true, because both sides of the equation are false.

3.1.7. Q.E.D.

Proof: By steps 3.1.5, 3.1.6 and 3.1.4, δ is as desired in step 3.1.3.

3.2. Q.E.D.

Proof:

$$\begin{aligned} \hat{u}(z) = \hat{u}(z) &\iff z^q = w^q \text{ and } z^q p(z) = w^q p(w) \\ &\iff z^q = w^q \text{ and } p(z) = p(w) \\ &\iff \exists l = 0, \ldots, q-1\colon w = ze^{2\pi i l/q} \text{ and } \hat{u}(z) = u'_l(z) \\ &\iff \exists l = 0, \ldots, q-1\colon w = ze^{2\pi i l/q} \text{ and } \hat{u} = u'_l, \end{aligned}$$

where in the last equivalence we used step 3.1.

4. m divides q. Let $k = q/m$.

Proof: Since $\hat{u} = u'_m$, then $\forall l \in \mathbb{Z}\colon \hat{u} = u'_{lm}$. Assume by contradiction that m does not divide q. Let

$$l^* = \max\{l \in \{1, \ldots, q-1\} \mid lm \in S\}.$$

Then $(l^* + 1)m - q$ is smaller than m and is in S. Contradiction.

5. Let

- for $l = 0, \ldots, k-1$,

$$\begin{aligned} \hat{\psi}_l\colon B_\delta(0) &\longrightarrow B_\delta(0) \\ z &\longmapsto ze^{2\pi i l/k}, \end{aligned}$$

- for $l = 0, \ldots, k-1$ $\hat{u}_l := \hat{u} \circ \hat{\psi}_l = \hat{u}$.

6. For all $z, w \in B_\delta(0)$, $\hat{u}(z) = \hat{u}(w)$ if and only if there exists some $l = 0, \ldots, k-1$ such that $w = ze^{2\pi i l/k}$ and $\hat{u} = u'_{lm}$.

Proof: By step 3, $\hat{u}(z) = \hat{u}(w)$ if and only if there exists some $l = 0, \ldots, q-1$ such that $w = ze^{2\pi i l/q}$ and $\hat{u} = u'_l$. Such an l in necessarily a multiple of m, $l = l'm$. Therefore, $l' \in \{0, \ldots, k-1\}$ is such that $w = ze^{2\pi i l'm/q} = ze^{2\pi i l'/k}$ and $\hat{u} = u^l_{l'm} = \hat{u}_{l'}$.

7. Let $\hat{v}$ be given by

$$\begin{aligned} \hat{v}\colon B_{\delta^k}(0) &\longrightarrow \psi(W) \\ z &\longmapsto \hat{u}(z^{1/k}), \end{aligned}$$

where $z^{1/k}$ is any w such that $w^k = z$. Then $\hat{v}$ is well defined, continuous and injective.

Proof:

7.1. $\hat{v}$ is well defined.

Proof: Let w_1, w_2 be such that $w_1^k = w_2^k = z$. Then, for some $l = 0, \dots, k-1$, $w_2 = w_1 e^{2\pi i l/k}$.

$$\begin{aligned} \hat{u}(w_2) &= \hat{u}(w_1 e^{2\pi i l/k}) \\ &= \hat{u}_l(w_1) \\ &= \hat{u}(w_1). \end{aligned}$$

7.2. $\hat{v}$ is continuous.

Proof:

7.2.1. $\hat{v}$ is continuous in $B_{\delta^k}(0)\backslash\{0\}$.

Proof: It suffices to assume that $z \in B_{\delta^k}(0)\backslash\{0\}$ and prove that $\hat{v}$ is continuous on a neighborhood of z. This last statement is true, because on a neighborhood of z we can define a continuous k-th root function $(\cdot)^{1/k}$, which will be such that $\hat{v} = \hat{u} \circ (\cdot)^{1/k}$.

7.2.2. $\hat{v}$ is continuous at 0.

Proof:

7.2.2.1. $\hat{v}(0) = 0$.

7.2.2.2. $\hat{v}|_{B_\delta(0)\backslash\{0\}} \colon B_\delta(0)\backslash\{0\} \longrightarrow \psi(W)$ is holomorphic.

Proof: At every point except 0, it is possible to define a k-th root holomorphic function $(\cdot)^{1/k}$ such that $\hat{v} = \hat{u} \circ (\cdot)^{1/k}$.

7.2.2.3. $\hat{v}$ is bounded on a neighborhood of 0.

Proof: By the polynomial representation for $\hat{u}$.

7.2.2.4. $\hat{v}$ admits a continuous extension to 0, $\hat{v}' \colon B_\delta(0) \longrightarrow \psi(W)$.

Proof: By steps 7.2.2.2, 7.2.2.3 and the Riemann removal of singularities theorem.

7.2.2.5. $\hat{v}'(0) = 0$.

Proof:

$$\begin{aligned} \hat{v}'(0) &= \lim_{z\to 0} \hat{v}'(z) && [\hat{v} \text{ is continuous}] \\ &= \lim_{\mathbb{R}^+ \ni t \to 0} \hat{v}'(t) && [\lim \text{ does not depend of the path}] \\ &= \lim_{\mathbb{R}^+ \ni t \to 0} \hat{v}(t) && [\text{on } \mathbb{R}^+,\ \hat{v} = \hat{v}'] \\ &= \lim_{\mathbb{R}^+ \ni t \to 0} \hat{u}(t^{1/k}) && [(\cdot)^{1/k} \text{ is a continuous function on } \mathbb{R}^+] \\ &= \hat{u}(0^{1/k}) && [\hat{u}, (\cdot)^{1/k} \text{ are continuous}] \\ &= 0. \end{aligned}$$

7.2.2.6. Q.E.D.

Proof: By steps 7.2.2.1, 7.2.2.5 and 7.2.2.4, $\hat{v} = \hat{v}'$. By step 7.2.2.4 $\hat{v} = \hat{v}'$ is continuous.

7.2.3. Q.E.D.

Proof: Steps 7.2.1 and 7.2.2.

7.3. $\hat{v}$ is injective.

Proof: It suffices to assume that z_1, z_2 are such that $\hat{v}(z_1) = \hat{v}(z_2)$ and to prove that $z_1 = z_2$. Let w_1, w_2 be such that $w_1^k = z_1$, $w_2^k = z_2$.

$$\begin{aligned} \hat{v}(z_1) = \hat{v}(z_2) &\Longrightarrow \hat{u}(w_1) = \hat{u}(w_2) \\ &\Longrightarrow \exists l \colon w_2 = w_1 e^{2\pi i l/k} && \text{[by step 6]} \\ &\Longrightarrow z_2 = w_2^k = w_1^k (e^{2\pi i l/k})^k = w_1^k = z_1. \end{aligned}$$

7.4. Q.E.D.

Proof: Steps 7.1, 7.2 and 7.3.

8. We may assume after shrinking δ that there exists $\Phi \colon (B_\delta(0), j) \longrightarrow (B, i)$ such that

- Φ is a biholomorphism,
- $\Phi(0) = 0$,
- $\Phi \circ \hat{\psi}_l \circ \Phi^{-1}(z) = e^{2\pi i l/k} z = \hat{\psi}_l(z) =: \tilde{\psi}_l(z)$.

Proof:

8.1. We may assume after shrinking δ that there exists U a neighborhood of 0 and $\Phi \colon (B_\delta(0), j) \longrightarrow (U, i)$ a biholomorphism such that $\Phi(0) = 0$.

Proof: $(B_\delta(0), j)$ is a two dimensional (over $\mathbb{R}$) almost complex manifold. By the Newlander-Nirenberg theorem, j is integrable. Therefore $(B_\delta(0), j)$ is a complex manifold. The map Φ is then just a complex coordinate chart.

8.2. We may assume that $U = B_1(0)$.

Proof: By composing Φ with a biholomorphism $U \longrightarrow B_1(0)$ coming from the Riemann mapping theorem. The map $U \longrightarrow B_1(0)$ maps 0 to 0.

8.3. $\hat{\psi}_l(z) = e^{2\pi i l/k}$.

Proof: The map

$$\begin{aligned} \mathbb{Z}_k &\longrightarrow \{\psi \in C^\infty(B_1(0), B_1(0)) \mid \psi(0) = 0, \psi \text{ is a biholomorphism}\} \\ l &\longmapsto \psi_l' \end{aligned}$$

is a group homomorphism and

$$\begin{aligned} &\{\psi \in C^\infty(B_1(0), B_1(0)) \mid \psi(0) = 0, \psi \text{ is a biholomorphism}\} \\ = &\{\psi \in C^\infty(B_1(0), B_1(0)) \mid \exists \theta \in [0, 2\pi) \colon \forall z \in B_1(0) \colon \psi(z) = z e^{i\theta}\}. \end{aligned}$$

8.4. Q.E.D.

Proof: Steps 8.1, 8.2 and 8.3.

9. Let

- $\Psi := \Phi^{-1}$,
- $\tilde{u} := \hat{u} \circ \Psi$,
- for each l, $\tilde{u}_l := \tilde{u} \circ \tilde{\psi}_l = \tilde{u}$.

10. Let $\tilde{v}$ be defined by

$$\tilde{v} \colon B_1(0) \longrightarrow \psi(W)$$

$$z \longmapsto \tilde{u}(z^{1/k}),$$

where $z^{1/k}$ is any w such that $w^k = z$. Then $\tilde{v}$ is well defined, and injective.

Proof:

10.1. $\tilde{v}$ is well defined.

Proof: It suffices to assume that w_1, w_2 are such that $w_1^k = w_2^k$ and prove that $\tilde{u}(w_1) = \tilde{u}(w_2)$. Since $w_1^k = w_2^k$, there exists some $l = 0, \ldots, k-1$ such that $w_2 = w_1 e^{2\pi i l/k}$.

$$\begin{aligned}
\tilde{u}(w_2) &= \hat{u} \circ \Psi(w_2) \\
&= \hat{u} \circ \Psi(w_1 e^{2\pi i l/k}) \\
&= \hat{u} \circ \Psi \circ \hat{\psi}_l(w_1) \\
&= \hat{u} \circ \hat{\psi}_l \circ \Psi(w_1) \\
&= \hat{u} \circ \Psi(w_1) \\
&= \tilde{u}(w_1).
\end{aligned}$$

10.2. $\tilde{v}$ is injective.

Proof: It suffices to assume that w_1, w_2 are such that $\tilde{u}(w_1) = \tilde{u}(w_2)$ and prove that $w_1^k = w_2^k$.

$$\begin{aligned}
\tilde{u}(w_1) = \tilde{u}(w_2) &\Longrightarrow \hat{u} \circ \Psi(w_1) = \hat{u} \circ \Psi(w_2) \\
&\Longrightarrow \exists l \colon \Psi(w_1) = \Psi(w_2) e^{2\pi i l/k} \\
&\Longrightarrow \Psi(w_1) = \Psi(w_2 e^{2\pi i l/k}) \\
&\Longrightarrow w_1 = w_2 e^{2\pi i l/k} \\
&\Longrightarrow w_1^k = w_2^k.
\end{aligned}$$

10.3. Q.E.D.

Proof: Steps 10.1 and 10.2.

11. $\tilde{v}$ is smooth and J-holomorphic.

Proof:

11.1. There exists a $p > 2$ such that $\tilde{v} \in W^{1,p}(B, \mathbb{C}^n)$.

Proof:

11.1.1. $\exists C > 0 \colon \forall z \in B_1(0) \colon |d\tilde{u}(z)| \leq C|z|^{q-1}$.

Proof: By the polynomial formula for $\hat{u}$.

11.1.2. $\forall z \in B_1(0) \colon |d\tilde{v}(z)| \leq \frac{C}{k}|z|^{m-1}$.

Proof:

$$\begin{aligned}
|d\tilde{v}(z)| &\leq |d\tilde{u}(z^{1/k})| \frac{1}{k} |z|^{\frac{1}{k}-1} \\
&\leq \frac{C}{k} |z|^{\frac{1}{k}(q-1)} |z|^{\frac{1}{k}-1} \\
&= \frac{C}{k} |z|^{m-1}.
\end{aligned}$$

11.1.3. $d\tilde{v}$ is of class L^p in $B_1(0)$ for any $p > 2$.

Proof: By step 11.1.2.

11.1.4. Q.E.D.

Proof: By step 11.1.2, $d\tilde{v}$ is the weak derivative of $\tilde{v}$.

11.2. $\tilde{v} \in C^\infty(B, \mathbb{C}^n)$.

Proof: By step 11.1 and elliptic regularity.

11.3. $\tilde{v}$ is holomorphic in $B_1(0)\backslash\{0\}$.

Proof: Everywhere except at 0 we can define a holomorphic map $(\cdot)^{1/k}$ such that $\tilde{v} = \tilde{u} \circ (\cdot)^{1/k}$.

11.4. $\tilde{v}$ is holomorphic at 0.

Proof: By steps 11.2 and 11.3.

11.5. Q.E.D.

Proof: Steps 11.2, 11.3 and 11.4.

12. Let $\varepsilon > 0$ such that $B_\varepsilon(0) \subset U$. Let $v = \psi^{-1} \circ \tilde{v}$, $\varphi = (\Psi \circ \phi)^k|_{B_\varepsilon(0)}$. Then v, φ are the desired maps.

Proof: φ is holomorphic, $\varphi(0) = (\Psi \circ \phi(0))^k = 0$, and v is injective because of step 10 and ψ being a biholomorphism. v is J-holomorphic by step 11.

13. Q.E.D.

Proof: By step 12. □

Define

$$
\begin{aligned}
\mathrm{C}(u) &\coloneqq \mathrm{CritPts}(u) \\
\mathrm{S}(u) &\coloneqq \mathrm{SelfInt}(u) \\
&\coloneqq \{z \in \Sigma \mid \exists w \in \Sigma\colon \\
&\qquad\qquad \exists U_z \text{ a neighborhood of } z\colon \\
&\qquad\qquad \exists U_w \text{ a neighborhood of } w\colon \\
&\qquad\qquad u(z) = u(w) \text{ and } u(U_z\backslash\{z\}) \cap u(U_w\backslash\{w\}) = \varnothing\} \\
\mathrm{CS}(u) &\coloneqq \mathrm{C}(u) \cup \mathrm{S}(u).
\end{aligned}
$$

Exercise 10.3. Let (Σ, j) be a closed connected Riemann surface, (M, J) be an almost complex manifold and $u\colon (\Sigma, j) \longrightarrow (M, J)$ be a nonconstant J-holomorphic curve. Then,

(i) There exists a closed connected Riemann surface Σ', a holomorphic map $\varphi\colon \Sigma \longrightarrow \Sigma'$, and a J-holomorphic curve $v\colon \Sigma' \longrightarrow M$ such that

- $u = v \circ \varphi$,
- $\deg \varphi \geq 1$,
- $\mathrm{CS}(v)$ is finite and $v|_{\mathrm{CS}(v)}\colon \Sigma'\backslash\mathrm{CS}(v) \longrightarrow M$ is an embedding.

(ii) If (Σ'', φ', v') are also as in (ii), then there exists a biholomorphism $\phi\colon \Sigma' \longrightarrow \Sigma''$ such that $v = v' \circ \phi$ and $\varphi' = \phi \circ \varphi$.

Proof sketch: See figure 4 below.

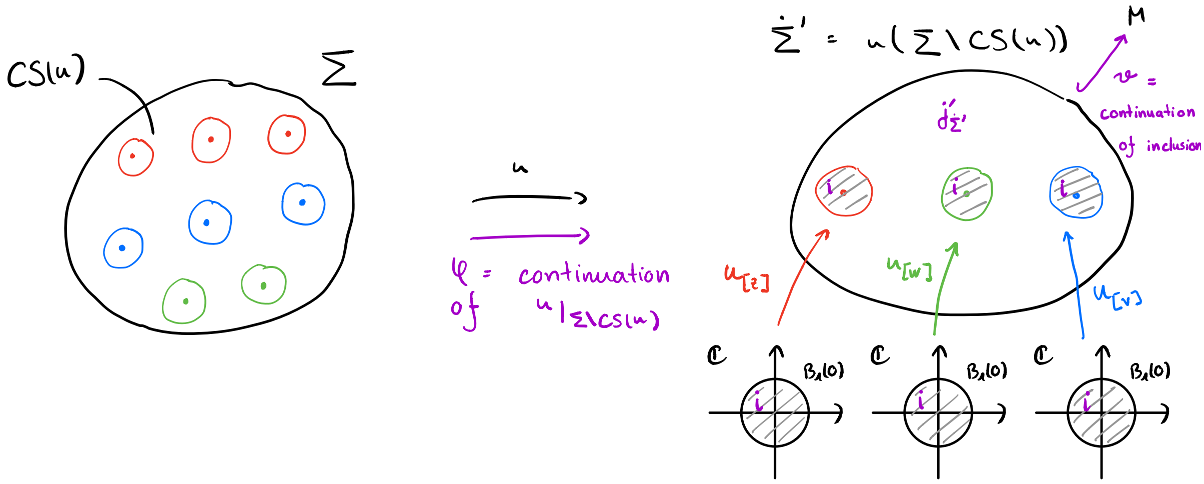


Figure 4: Proof sketch of exercise 10.3.

Solution:

1. (i):

   Proof:

   1.1. $\mathrm{CS}(u)$ is a finite set.

   Proof:

   1.1.1. $\mathrm{C}(u)$ is a finite set.

   Proof: Since Σ is compact, it suffices to show that it is discrete. One can conclude that it is discrete using the coordinates given by the Micallef-White theorem.

   1.1.2. $\mathrm{S}(u)$ is a finite set.

   Proof: By theorem E.1.2. in [MS04], which is a consequence of the Micallef-White theorem.

   1.1.3. Q.E.D.

   Proof: By steps 1.1.1 and 1.1.2.

   1.2. Let $\dot{\Sigma}' := u(\Sigma \setminus \mathrm{CS}(u)) \subset M$. Then,

   (i) $\dot{\Sigma}'$ is a smooth submanifold of M;

   (ii) $\forall z \in \Sigma \setminus \mathrm{CS}(u)\colon T_{u(z)}\dot{\Sigma}' = du(z) \cdot T_z\Sigma$;

   (iii) $\forall p \in \dot{\Sigma}'\colon J_p(T_p\dot{\Sigma}') \subset T_p\dot{\Sigma}'$.

   Let $j'_{\dot{\Sigma}'} := J|_{\dot{\Sigma}'} \in C^\infty(\dot{\Sigma}', \mathrm{Hom}_{\mathbb{R}}(T\dot{\Sigma}', T\dot{\Sigma}'))$.

   Proof:

   1.2.1. $\dot{\Sigma}'$ is a smooth submanifold of M;

   1.2.2. $\forall z \in \Sigma \setminus \mathrm{CS}(u)\colon T_{u(z)}\dot{\Sigma}' = du(z) \cdot T_z\Sigma$;

   Proof:

$$T_{u(z)}\dot{\Sigma}' = \{\dot{\gamma}(0) \mid \gamma\colon I \longrightarrow \dot{\Sigma}' \text{ is } C^\infty, \gamma(0) = u(z)\}$$

$$
\begin{aligned}
&= \left\{ \frac{d}{dt}(u \circ \rho)(0) \,\middle|\, \rho\colon I \longrightarrow \Sigma \setminus \mathrm{CS}(u) \text{ is } C^\infty, \rho(0) = z \right\} \\
&= \{ du(z) \cdot \dot\rho(0) \mid \rho\colon I \longrightarrow \Sigma \setminus \mathrm{CS}(u) \text{ is } C^\infty, \rho(0) = z \} \\
&= du(z) \cdot \{ \dot\rho(0) \mid \rho\colon I \longrightarrow \Sigma \setminus \mathrm{CS}(u) \text{ is } C^\infty, \rho(0) = z \} \\
&= du(z) \cdot T_z\Sigma.
\end{aligned}
$$

1.2.3. $\forall p \in \dot\Sigma'\colon J_p(T_p\dot\Sigma') \subset T_p\dot\Sigma'$.

Proof: It suffices to assume that $p \in \dot\Sigma'$ and prove that $J_p(T_p\dot\Sigma') \subset T_p\dot\Sigma'$. There exists $z \in \Sigma$ such that $u(z) = p$.

$$
\begin{aligned}
J_p(T_p\dot\Sigma') &= J_p \circ du(z) \cdot T_z\Sigma \\
&= du(z) \circ j(z) \cdot T_z\Sigma \\
&\subset du(z) \cdot T_z\Sigma \\
&= T_p\dot\Sigma'.
\end{aligned}
$$

1.2.4. Q.E.D.

Proof: Steps 1.2.1, 1.2.2 and 1.2.3.

1.3. Let $\sim$ be the equivalence relation in $\mathrm{CS}(u)$ defined by $\forall z, w \in \mathrm{CS}(u)\colon z \sim w$ if and only if $u(z) = u(w)$ and there exist $U_z \subset \Sigma \setminus \mathrm{CS}(u)$ a neighborhood of z, $U_w \subset \Sigma \setminus \mathrm{CS}(u)$ a neighborhood of z and $\phi\colon U_z \longrightarrow U_w$ a biholomorphism such that $u|_{U_z} = u \circ \phi$ and $U_z \cap U_w = \varnothing$. Let $\widetilde{\mathrm{CS}(u)} = \mathrm{CS}(u)/\sim$.

1.4. There exist maps $u_z\colon B_1(0) \longrightarrow M$ and neighborhoods U_z of z, for each $z \in \Sigma$, such that

(i) u_z is J-holomorphic;

(ii) u_z is injective;

(iii) $\operatorname{im} u_z = u(U_z)$;

(iv) $z \sim w \Longrightarrow u_z = u_w$ and $u(U_z) = u(U_w)$.

Let $u_{[z]} \coloneqq u_z$ and $V_{[z]} \coloneqq u(U_z)$.

Proof: Let $z \in \mathrm{CS}(u)$. Let

$$
\begin{aligned}
z \in U \subset (\Sigma, j), &\qquad U' \subset (\mathbb{C}, i), \quad \phi\colon U \longrightarrow U', \\
u(z) \in V \subset (M, J), &\qquad V' \subset (\mathbb{C}^n, J'), \quad \psi\colon V \longrightarrow V'
\end{aligned}
$$

be complex coordinate charts around z, $u(z)$ such that $u(U) \subset V$. By the Riemann mapping theorem, we can assume that $U' = B_1(0)$. Let $\hat u$ be the local representative of u with respect to these coordinates. By exercise 10.2, there exist $\varphi\colon B_\varepsilon \longrightarrow B$ holomorphic and $v\colon B \longrightarrow \mathbb{C}^n$ injective and J-holomorphic such that $\hat u|_{B_\varepsilon} = v \circ \varphi$. By the definition of $\sim$, we may assume that if $z \sim w$ then the local representatives of u with respect to coordinate charts centred at z and w coincide. Restricting ε and changing the domain of v, $B_1(0)$, to a smaller neighborhood of 0, $U'' \subset B_1(0)$, we may assume that φ is surjective. By the Riemann mapping theorem we may assume that $U'' = B_1(0)$. Then, $u_z \coloneqq \psi^{-1} \circ v$ and $U_z \coloneqq \phi^{-1} B_\varepsilon(0)$ are as desired. The above proves that there exist maps u_z satisfying (i), (ii), (iii) and (iv).

1.5. Let

- $\Phi := \coprod_{[z]\in\widetilde{\mathrm{CS}(u)}} u_{[z]}|_{B_1(0)\setminus\{0\}} \colon \coprod_{[z]\in\widetilde{\mathrm{CS}(u)}} B_1(0)\setminus\{0\} \longrightarrow \dot{\Sigma}'$,
- $\Sigma' := \dot{\Sigma}' \cup_\Phi \left(\coprod_{[z]\in\widetilde{\mathrm{CS}(u)}} B_1(0)\right)$, where $\cup_\Phi$ means "gluing with respect to the map Φ",
- j' be given by $j'_{\dot{\Sigma}'}$ on $\dot{\Sigma}'$, i on $B_1(0)$.

Then (Σ', j') is a compact Riemann surface.

1.6. Let $v \colon \Sigma' \longrightarrow M$ be given by the inclusion $i_{\Sigma' \subset M}$ on $\dot{\Sigma}'$, Φ on $\coprod_{[z]\in\widetilde{\mathrm{CS}(u)}} B_1(0)$. Then,
   (i) $v \colon (\Sigma', j') \longrightarrow (M, J)$ is pseudoholomorphic;
   (ii) $v|_{\dot{\Sigma}'}$ is an embedding.

1.7. Let $\dot{\varphi} := u|_{\Sigma\setminus\mathrm{CS}(u)} \colon \Sigma \setminus \mathrm{CS}(u) \longrightarrow \dot{\Sigma}'$. Then, there exists a unique $\varphi \colon \Sigma \longrightarrow \Sigma'$ continuous such that $\varphi|_{\Sigma\setminus\mathrm{CS}(u)} = \dot{\varphi}$. φ also satisfies
   (i) φ is holomorphic;
   (ii) $u = v \circ \varphi$.

2. (ii)

Proof: It suffices to assume that for $j = 1, 2$, $(\Sigma'_j, \varphi_j, v_j)$ is a set of data as in (i), and prove that there exists a biholomorphism $\phi \colon \Sigma_1 \longrightarrow \Sigma_2$ such that $v_1 = v_2 \circ \phi$ and $\varphi_2 = \phi \circ \varphi_1$. Then, for $j = 1, 2$

$$v_j|_{\Sigma'_j\setminus\mathrm{CS}(v_j)} \colon \Sigma'_j \setminus \mathrm{CS}(v_j) \longrightarrow M$$

is an embedding. Consider the maps

$$\dot{v}_1 := v_1|_{\Sigma'_1\setminus(\mathrm{CS}(v_1)\cup v_1^{-1}(v_2(\mathrm{CS}(v_2))))} \colon \Sigma'_1 \setminus (\mathrm{CS}(v_1) \cup v_1^{-1}(v_2(\mathrm{CS}(v_2)))) \longrightarrow M$$

$$\dot{v}_2 := v_2|_{\Sigma'_2\setminus(\mathrm{CS}(v_2)\cup v_2^{-1}(v_1(\mathrm{CS}(v_1))))} \colon \Sigma'_2 \setminus (\mathrm{CS}(v_2) \cup v_2^{-1}(v_1(\mathrm{CS}(v_1)))) \longrightarrow M.$$

Then, $\dot{v}_2^{-1} \circ \dot{v}_1$ is bijective, holomorphic and an embedding. There exists a unique continuous map $\phi \colon \Sigma'_1 \longrightarrow \Sigma'_2$ that extends $\dot{v}_2^{-1} \circ \dot{v}_1$. The map ϕ is as desired.

3. Q.E.D.

Proof: Steps 1 and 2. ☐

11 Exercise sheet No. 11 - 10-07-2019

In this and the next exercise sheet we are going to prove positivity of intersections. We follow the presentation in [MS04].

Exercise 11.1. Let (M, J) be an almost complex manifold, (Σ, j) be a closed Riemann surface and $u\colon \Sigma \longrightarrow M$ be a simple J-holomorphic curve. Then the set

$$\begin{aligned} Z &:= \{z \in \Sigma \mid du(z) = 0 \text{ or } \#u^{-1}(u(z)) > 1\} \\ &= \{z \in \Sigma \mid z \text{ is a noninjective point of } u\} \end{aligned}$$

is finite.

Proof:

1. It suffices to show that

    (i) There exists U a neighborhood of the diagonal $\Delta \subset \Sigma \times \Sigma$ such that $\forall (z_0, z_1) \in U\colon (u(z_0) = u(z_1) \Longrightarrow z_0 = z_1)$.

    (ii) $S(u) := \{(z_0, z_1) \in \Sigma \times \Sigma \mid u(z_0) = u(z_1), z_0 \neq z_1\}$ is a discrete set.

    Proof:

    1.1. $S(u) \subset \Sigma \times \Sigma \setminus U$.

    Proof:

    1.1.1. It suffices to assume that $z_0, z_1 \in \Sigma$, $u(z_0) = u(z_1)$, $z_0 = z_1$, and prove that $z_0, z_1 \notin U$.

    1.1.2. Assume by contradiction that $(z_0, z_1) \in U$.

    1.1.3. Q.E.D.

    Proof:

    $$\begin{aligned} z_0 &= z_1 \quad [\text{by } (z_0, z_1) \in U \text{ and definition of } U] \\ &\neq z_0 \quad [\text{by assumption in step 1.1.1}]. \end{aligned}$$

    1.2. $\Sigma \times \Sigma \setminus U$ is compact.

    Proof: $\Sigma \times \Sigma \setminus U$ is a closed subset of the compact topological space $\Sigma \times \Sigma$.

    1.3. $S(u)$ is finite.

    Proof: By step 1.1 $S(u) \subset \Sigma \times \Sigma$, by step 1.2 $\Sigma \times \Sigma \setminus U$ is compact, and by assumption $S(u)$ is discrete.

    1.4. $C(u)$ is finite.

    Proof: Lemma 2.4.1 in [MS04].

    1.5. Q.E.D.

    Proof: $Z(u) = \pi_1 S(u) \cup C(u)$. By steps 1.3 and 1.4 this is a finite set.

2. (i):

    Proof:

    2.1. It suffices to show that $\forall z_0 \in \Sigma\colon \exists U_z$ a neighborhood of $z\colon u|_{U_z}\colon U_z \longrightarrow M$ is injective.

Proof:

2.1.1. For each $z \in \Sigma$, let U_z be a neighborhood of z such that $u|_{U_z} \colon U_z \longrightarrow M$ is injective.

2.1.2. $\{U_z \times U_z\}_{z\in\Sigma}$ is an open covering of Δ.

2.1.3. Let $U \coloneqq \bigcup_{z\in\Sigma} U_z \times U_z$.

2.1.4. $\forall (z_0, z_1) \in U \colon (u(z_0) = u(z_1) \Longrightarrow z_0 = z_1)$.

Proof:

2.1.4.1. It suffices to assume that $(z_0, z_1) \in U$, $u(z_0), u(z_1)$ and prove that $z_0 = z_1$.

2.1.4.2. $\exists z \in \Sigma \colon (z_0, z_1) \in U_z \times U_z$.

2.1.4.3. z_0, z_1.

Proof: Since $u(z_0) = u(z_1)$, by step 2.1.4.2 $z_0, z_1 \in U_z$, and by step 2.1.1 $u|_{U_z}$ is injective.

2.1.4.4. Q.E.D.

Proof: Steps 2.1.4.1 and 2.1.4.3.

2.1.5. Q.E.D.

Proof: Step 2.1.4.

2.2. We may assume that $du(z_0) = 0$.

Proof: If $du(z_0) \neq 0$, the result follows by the inverse function theorem.

2.3. We may assume that $z_0 = 0$, $\Sigma = B_1(0)$, $M = \mathbb{C}^n$ and
$$\begin{aligned} u \colon B_1(0) &\longrightarrow \mathbb{C}^n \\ z &\longmapsto (z^k, z^k p(z)), \end{aligned}$$
where $k \in \mathbb{N}$ and $p \colon \mathbb{C} \longrightarrow \mathbb{C}^{n-1}$ is a polynomial such that $p(0) = 0$.

Proof: It suffices to show that there exists
$$\begin{aligned} u' \colon B_1(0) &\longrightarrow \mathbb{C}^n \\ z &\longmapsto (z^k, z^k p(z)), \end{aligned}$$
where $k \in \mathbb{N}$ and $p \colon \mathbb{C} \longrightarrow \mathbb{C}^{n-1}$ is a polynomial such that $p(0) = 0$, such that if the result holds for u', i.e.:

- there exists a neighborhood U_0' of 0 in $B_1(0)$ such that $u'|_{U_0}$ is injective

then the result holds for u, i.e.:

- there exists a neighborhood U_0 of 0 in Σ such that $u|_{U_0}$ is injective.

Such a $u' \colon B_1(0) \longrightarrow \mathbb{C}^n$ exists by the Micallef-White theorem.

2.4. There exists U_0 a neighborhood of 0 such that $u|_{U_0} \colon U_0 \longrightarrow \mathbb{C}^n$ is injective.

Proof:

2.4.1. Assume by contradiction that $\forall U$ a neighborhood of $0 \colon \exists z, w \in U \colon z \neq w$ and $u(z) = u(w)$.

2.4.2. There exist sequences $(z_\nu), (w_\nu)$ in $B_1(0)$ such that $z_\nu \longrightarrow 0$, $w_\nu \longrightarrow 0$, $z_\nu \neq w_\nu$, $u(z_\nu) = u(w_\nu)$, $z_\nu \neq 0$, $w_\nu \neq 0$.

Proof: By taking z_ν, w_ν to be $z.w$ from step 2.4.1 associated to $U =$

$B_{1/\nu}(0)$.

2.4.3. $\forall \nu \in \mathbb{N}\colon z_\nu^k = w_\nu^k$ and $p(z_\nu) = p(w_\nu)$.

Proof: By $u(z_\nu) = u(w_\nu)$ (step 2.4.2) and the polynomial representation of u in step 2.3.

2.4.4. We may assume that $\exists \zeta \in \mathbb{C}\colon \zeta \neq 1, \zeta^k = 1, w_k = \zeta z_k$.

Proof: Let $\zeta_\nu = w_\nu / z_\nu$. Then $\zeta_\nu^k = 1$ and $\zeta_\nu \neq 1$. Since $\zeta_\nu^k = 1$, then $\zeta_\nu = e^{2\pi i l_\nu / k}$ for some $l_\nu = 0, \dots k-1$. For some $l = 0, \dots, k-1$, the set $\{\nu \in \mathbb{N} \mid l = l_\nu\}$ is infinite. Take a subsequence corresponding to that subset of $\mathbb{N}$. Then ζ_ν is constant in that subsequence.

2.4.5. The polynomial $z \longmapsto p(z) - p(\zeta z)$ is zero.

Proof: Since $p(z_\nu) = p(w_\nu) = p(\zeta z_\nu)$, the polynomial $z \longmapsto p(z) - p(\zeta z)$ is zero at an infinite number of distinct points.

2.4.6. $\forall z \in B_1(0)\colon u(\zeta z) = u(z)$.

Proof:

$$\begin{aligned} u(\zeta z) &= (\zeta^k z^k, \zeta^k z^k p(\zeta z)) \\ &= (z^k, z^k p(z)) \\ &= u(z). \end{aligned}$$

2.4.7. Q.E.D.

Proof: By assumption, u is simple. By step 2.4.6, it is not simple. Contradiction.

2.5. Q.E.D.

Proof: Steps 2.1 to 2.4.

3. (ii)

Proof:

3.1. It suffices to assume that $z_0, z_1 \in \Sigma$, $z_0 \neq z_1$, $u(z_0) = u(z_1) =: x$, and prove that there exists U a neighborhood of (z_0, z_1) in $\Sigma \times \Sigma$ such that $U \cap S(u) = \{(z_0, z_1)\}$.

3.2. There exist C^1-diffeomorphisms in the image

$$\begin{aligned} \varphi_i &\colon (U_i, z_i) \longrightarrow (\mathbb{C}, 0), \\ \psi &\colon (W, x) \longrightarrow (\mathbb{C}^n, 0), \end{aligned}$$

for $i = 1, 2$, such that the maps $u_i := \psi \circ u \circ \varphi_i^{-1}$ are of the form $u_0(z) = z^k(a + p(z))$, $u_1(z) = z^l(b + q(z))$, $p(0) = q(0) = 0$, where $a, b \in \mathbb{C}^n \setminus \{0\}$, $k, l \in \mathbb{N}$, and $p, q\colon \mathbb{C} \longrightarrow \mathbb{C}^n$ are polynomials. There exist unitary matrices $L_0, L_1 \in \mathbb{C}^{n \times n}$ such that

- $L_0 u_0(z) = (z^k, z^k p'(z))$, where p' is a polynomial such that $p'(0) = 0$;
- $L_1 u_1(z) = (z^k, z^k q'(z))$, where q' is a polynomial such that $q'(0) = 0$;
- $L_0^{-1}(\mathbb{C} \times \{0\}) = L_1^{-1}(\mathbb{C} \times \{0\}) \Longrightarrow L_0 = L_1$.

Proof: By the Micallef-White theorem.

3.3. It suffices to show that there exist $U_0' \subset \varphi_0(U_0)$ a neighborhood of 0 in $\mathbb{C}$,

$U_1' \subset \varphi_1(U_1)$ a neighborhood of 0 in $\mathbb{C}$ such that for all $z \in U_0'$, $w \in U_1'$

$$z \neq w, u_0(z) = u_1(w) \Longrightarrow z = w = 0.$$

Proof: If such neighborhoods U_0', U_1' exist, then $U := \varphi_0^{-1}(U_0') \times \varphi_1^{-1}(U_1')$ is as desired in step 3.1.

3.4. Case: a, b are linearly independent.

Proof:

3.4.1. $\exists \delta > 0\colon \forall \lambda, \mu \in \mathbb{C}\colon |\lambda a + \mu b| \geq \delta(|\lambda| + |\mu|)$.

Proof: a, b are linearly independent.

3.4.2. $\exists c > 0\colon \forall z \in B\colon |p(z)| \leq c|z|$ and $|q(z)| \leq c|z|$.

Proof: Since $p(0) = 0$ and $q(0) = 0$, there exist unique polynomials p'', q'' such that $p(z) = zp''(z)$ and $q(z) = zq''(z)$. Let

$$c := \max\left\{ \max_{z \in \overline{B_1(0)}} |p''(z)|,\ \max_{z \in \overline{B_1(0)}} |q''(z)| \right\}.$$

3.4.3. $\forall z, w \in \mathbb{C}\colon |u_0(z) - u_1(w)| \geq |z|^k(\delta - c|z|) + |w|^k(\delta - c|w|)$.

Proof:

$$\begin{aligned}
&|u_0(z) - u_1(w)| \\
&\quad = |z^k(a + p(z)) - w^l(b + q(w))| && \text{[def. of } u_0, u_1 \text{ in step 3.2]} \\
&\quad = |z^k a - w^l b + z^k p(z) - w^l q(w)| \\
&\quad \geq |z^k a - w^l b| - |z^k p(z) - w^l q(w)| \\
&\quad \geq \delta(|z|^k + |w|^l) - |z^k p(z)| - |w^l q(w)| && \text{[step 3.4.1]} \\
&\quad = \delta|z|^k + \delta|w|^l - |z|^k|p(z)| - |w|^l|q(w)| \\
&\quad \geq \delta|z|^k + \delta|w|^l - |z|^k c|z| - |w|^l c|w| && \text{[step 3.4.2]} \\
&\quad = |z|^k(\delta - c|z|) + |w|^l(\delta - c|w|).
\end{aligned}$$

3.4.4. $\forall z, w \in B_{\delta/c}(0)\colon (|u_0(z) - u_1(w)| = 0 \Longleftrightarrow z = w = 0)$.

Proof:

3.4.4.1. $(\Longleftarrow)$:

Proof: $z = 0, w = 0 \Longrightarrow u_0(z) = 0, u_1(w) = 0$.

3.4.4.2. $(\Longrightarrow)$:

Proof:

$$\begin{aligned}
&0 = |u_0(z) - u_1(w)| \\
&\quad \geq |z|^k(\delta - c|z|) + |w|^l(\delta - c|w|) && \text{[step 3.4.3]} \\
&\quad \geq 0 \\
\Longrightarrow\ &0 = |z|^k(\delta - c|z|) + |w|^l(\delta - c|w|) \\
\Longrightarrow\ &0 = |z|^k(\delta - c|z|) && \text{[each term is positive} \\
&0 = |w|^l(\delta - c|w|) && \text{and the sum is 0]} \\
\Longrightarrow\ &0 = z, w && [z, w \in B_{\delta/c}(0)].
\end{aligned}$$

3.4.4.3. Q.E.D.

Proof: Steps 3.4.4.1 and 3.4.4.2.

3.4.5. Q.E.D.

Proof: By step 3.4.4, $U'_0 = U'_1 = B_{\delta/c}(0)$ is the desired open set in step 3.3.

3.5. Case: a, b are linearly dependent.

Proof:

3.5.1. We may assume that $a = b = (1, 0) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and that $\forall z\colon p(z), q(z) \in \{0\} \times \mathbb{C}^{n-1}$.

Proof: Since a, b are linearly dependent, the matrices L_0, L_1 from the Micallef-White theorem in step 3.1 are equal. Replace u_0, u_1 by $u'_j = Lu_j$ and ψ by $\psi' = L\psi$. If the result is true for u'_j then it is true for u_j.

3.5.2. Assume by contradiction that for all $U'_0 \subset \varphi_0(U_0)$ a neighborhood of 0 in $\mathbb{C}$, $U'_1 \subset \varphi_1(U_1)$ a neighborhood of 0 in $\mathbb{C}$, there exist $z \in U'_0$, $w \in U'_1$ such that

$$z \neq w, u_0(z) = u_1(w), (z \neq 0 \text{ or } w \neq 0).$$

3.5.3. There exist sequences $(z'_\nu), (w'_\nu)$ such that

- $\forall \nu\colon z'_\nu \neq 0, w'_\nu \neq 0$;
- $\forall \nu\colon u_0(z'_\nu) = u_1(w'_\nu)$;
- $\forall \nu\colon z'_\nu \neq 0, w'_\nu \neq 0$.

Proof: For each $\nu \in \mathbb{N}$, let z'_ν, w'_ν be the z, w associated to $U'_0 = B_{1/\nu}$, $U'_1 = B_{1/\nu}$ from step 3.5.2. Since

$$\begin{aligned}(z'^k_\nu, z'^k_\nu p(z'_\nu)) &= u_0(z'_\nu) \\ &= u_1(w'_\nu) \\ &= (w'^l_\nu, w'^l_\nu q(w'_\nu))\end{aligned}$$

and since one of the numbers z'_ν, w'_ν is nonzero, the other one must be nonzero as well.

3.5.4. There exist sequences (z_ν), (w_ν) such that

- $\forall \nu\colon z_\nu \neq 0, w_\nu \neq 0$;
- $\forall \nu\colon u_0(z^l_\nu) = u_1(w^k_\nu)$;
- $\forall \nu\colon z_\nu \neq 0, w_\nu \neq 0$.

Proof: For each ν, let z_ν be such that $z^l_\nu = z'_\nu$ and let w_ν be such that $w^k_\nu = w'_\nu$.

3.5.5. Let $m := kl$. We may assume (after passing to a subsequence) that there exists $\zeta \in \mathbb{C} \setminus \{1\}$ such that $\zeta^m = 1$ and $w_\nu = \zeta z_\nu$ for all ν.

Proof: Let $\zeta_\nu = w_\nu / z_\nu$. Then $\zeta^m_\nu = 1$ and $\zeta_\nu \neq 1$. Since $\zeta^m_\nu = 1$, then $\zeta_\nu = e^{2\pi i l_\nu / m}$ for some $l_\nu = 0, \dots m-1$. For some $l = 0, \dots, m-1$, the set $\{\nu \in \mathbb{N} \mid l = l_\nu\}$ is infinite. Take a subsequence corresponding to that subset of $\mathbb{N}$. Then ζ_ν is constant in that subsequence.

3.5.6. The polynomial $z \longmapsto p(z^l) - q(\zeta^k z^k)$ is zero.

Proof: Since $p(z^l_\nu) = q(w^k_\nu) = p(\zeta^k z^k_\nu)$, this polynomial has infinitely many zeros.

3.5.7. $\forall z \in \overline{B_1(0)}\colon u_0(z^l) = u_1(\zeta^k z^k)$.

Proof:

$$\begin{aligned} u_1(\zeta^k z^k) &= (\zeta^{lk} z^{lk}, \zeta^{lk} z^{lk} q(\zeta^k z^k)) \\ &= (z^{lk}, z^{lk} p(z^k)) \\ &= u_0(z^l). \end{aligned}$$

3.5.8. Q.E.D.

Proof: By assumption, u is simple. By step 3.5.7, it is not simple. Contradiction.

3.6. Q.E.D.

Proof: Steps 3.3, 3.4 and 3.5.

4. Q.E.D.

Proof: Steps 1, 2 and 3. □

Exercise 11.2. Let $u\colon \overline{B_1(0)} \longrightarrow \mathbb{C} \times \mathbb{C}^{n-1}$ be a polynomial that is not multiply covered of the form $u(z) = z^k(1, p(z))$ for $k \in \mathbb{N}$ and $p(0) = 0$. Let $U \subset \mathbb{C}$ be a neighborhood of 0 and $V \subset \mathbb{C}$, $W \subset \mathbb{C}^n$ be closed subsets containing 0 such that $V \subset U$, $u(V) \subset \operatorname{int}(W)$, $W \cap u(\partial U) = \varnothing$. Assume that the sets U, V, W are balls centred at 0. Then, for all $\varepsilon > 0$ there exists an immersion $v\colon U \longrightarrow \mathbb{C}^n$ such that:

(i) $v|_{U \setminus u^{-1}(W)} = u|_{U \setminus u^{-1}(W)}$ and $v(U \cap u^{-1}(W)) \subset W$;

(ii) v is holomorphic on V and $\|u - v\|_{C^1} < \varepsilon$;

(iii) $\#v^{-1}(0) = k$.

Proof:

1. For each $\delta > 0$, define

$$\begin{aligned} f_\delta(z) &:= \prod_{j=0}^{k-1} (z + j\delta) \\ u_\delta(z) &:= f_\delta(z)(1, p(z)). \end{aligned}$$

2. There exists U_1 a neighborhood of 0 in $\mathbb{C}$ such that

$$\forall z \in U_1 \setminus \{0\}\colon \forall \lambda \in \mathbb{C}\colon \lambda^k = 1, \lambda \neq 1 \Longrightarrow p'(z) \neq 0, p(z) \neq p(\lambda z).$$

Proof: $U_1 = (p')^{-1}(\mathbb{C}^{n-1} \setminus \{0\}) \cap \bigcap_{j=1}^k \{z \in \mathbb{C} \mid p(z) \neq p(e^{2\pi i j/k} z)\}$ is such an open set.

3. $\forall \delta > 0\colon u_\delta|_{U_1}$ is an immersion.

Proof:

3.1. It suffices to assume that $z \in U_1$, and prove that $u_\delta'(z) \neq 0$.

3.2. Assume by contradiction that $u_\delta'(z) = 0$.

3.3. $f_\delta'(z) = 0$ and $f_\delta(z) = 0$.

Proof:

$$\begin{aligned} 0 &= u_\delta'(z) \\ &= f_\delta'(z)(1, p(z)) + f_\delta(z)(0, p'(z)) \\ &= (f_\delta'(z), f_\delta'(z) p(z) + f_\delta(z) p'(z)) \end{aligned}$$

$$\Longrightarrow \begin{cases} f'_\delta(z) = 0 \\ f'_\delta(z)p(z) + f_\delta(z)p'(z) = 0 \end{cases}$$
$$\Longrightarrow \begin{cases} f'_\delta(z) = 0 \\ f_\delta(z)p'(z) = 0 \end{cases}$$
$$\Longrightarrow \begin{cases} f'_\delta(z) = 0 \\ f_\delta(z) = 0. \end{cases}$$

3.4. Q.E.D.

Proof: By step 3.3, $f'_\delta(z) = f_\delta(z) = 0$. By definition of f_δ in step 1, no zero of f_δ has multiplicity 2. Contradiction.

4. $\forall \delta \in (0, \varepsilon)\colon u_\delta^{-1}(0) = \{j\delta \mid j = 0, \dots, k-1\}$.

Proof:
$$\begin{aligned} u_\delta^{-1}(0) &= \{z \in \mathbb{C} \mid u_\delta(z) = 0\} \\ &= \{z \in \mathbb{C} \mid f_\delta(z)(1, p(z)) = 0\} \\ &= \{z \in \mathbb{C} \mid f_\delta(z) = 0\} \\ &= \{j\delta \mid j = 0, \dots, k-1\} \qquad \text{[def. of } f_\delta \text{ in step 1].} \end{aligned}$$

5. Let V' be an open set such that $V \subset V' \subset U$ and $\bar{V}' \subset \operatorname{int}(u^{-1}(W))$.

Proof: Since
$$\begin{aligned} V &\subset u^{-1}(u(V)) \\ &\subset u^{-1}(\operatorname{int}(W)) \quad \text{[by hypothesis]} \\ &\subset \operatorname{int} u^{-1}(W) \quad [u \text{ is continuous}] \end{aligned}$$
then $V \subset U \cap \operatorname{int} u^{-1}(W)$. Let V' be any open set such that $V \subset V' \subset U \cap \operatorname{int} u^{-1}(W)$.

6. Let $\beta\colon U \longrightarrow [0,1]$ be a smooth function such that $\beta(z) = 1$ if $z \in V$ and $\beta(z) = 0$ if $z \in U \setminus V'$.

7. For each $\delta > 0$, define $v_\delta(z) \coloneqq u(z) + \beta(z)(u_\delta - u(z))$.

8. $\exists \delta_1 > 0\colon \forall \delta' \in (0, \delta_1)\colon v_\delta$ is an immersion.

Proof: On U_1, $v_\delta = u_\delta$ which is an immersion. On $U \setminus V'$, $v_\delta = u$ which is an immersion. On $V' - U_1$, u is an immersion. Being an immersion is an open condition. Choose δ so small that v_δ so C^1-close to u that v_δ is still an immersion.

9. $\forall \delta > 0\colon v_\delta|_{U \setminus u^{-1}(W)} = u|_{U \setminus u^{-1}(W)}$.

Proof: By definition of v_δ in step 7, def. of β in step 6, and since $V' \subset u^{-1}(W)$ (step 5), $U \setminus u^{-1}(W) \subset U \setminus V$.

10. $\exists \delta_3 > 0\colon \forall \delta \in (0, \delta_3)\colon v_\delta(U \cap u^{-1}(W)) \subset W$.

Proof: Because $u(U \cap u^{-1}(W)) \subset W$, $W \cap u(\partial U) = \varnothing$ and $\delta \longmapsto v_\delta$ is continuous.

11. $\forall \delta > 0\colon v_\delta$ is holomorphic in V.

Proof: In V, $v_\delta = u_\delta$ is holomorphic.

12. $\exists \delta_4 > 0\colon \forall \delta \in (0, \delta_4)\colon \|v_\delta - u\|_{C^1} < \varepsilon$.

Proof:

$$\begin{aligned}\|v_\delta - u\|_{C^1} &= \sup_{z\in U}|v_\delta - u(z)| + \sup_{z\in U}|v_\delta' - u'(z)| \\ &= \sup_{z\in U}|\beta(z)(u_\delta(z) - u(z))| \\ &\quad + \sup_{z\in U}|\beta'(z)(u_\delta(z) - u(z)) + \beta(z)(u_\delta'(z) - u'(z))|.\end{aligned}$$

13. $\exists \delta_5 > 0\colon \forall \delta \in (0, \delta_5)\colon \# v_\delta^{-1}(0) = k$.

   Proof:

$$\begin{aligned}v_\delta^{-1}(0) &= \{z \in U \mid v_\delta(z) = 0\} \\ &= \{z \in V \mid v_\delta(z) = 0\} \cup \{z \in U \setminus V' \mid v_\delta(z) = 0\} \cup \{z \in V' \setminus V \mid v_\delta(z) = 0\} \\ &= \{z \in V \mid u_\delta(z) = 0\} \cup \{z \in U \setminus V' \mid u(z) = 0\} \cup \{z \in V' \setminus V \mid v_\delta(z) = 0\} \\ &= \big(V \cap \{j\delta \mid j = 0, \dots, k-1\}\big) \cup \big(U \setminus V' \cap \{0\}\big) \cup \{z \in V' \setminus V \mid v_\delta(z) = 0\} \\ &= \big(V \cap \{j\delta \mid j = 0, \dots, k-1\}\big) \cup \{z \in V' \setminus V \mid v_\delta(z) = 0\}.\end{aligned}$$

   Choose δ_5 so small that $\{j\delta \mid j = 0, \dots, k-1\} \subset V$, $\{z \in V' \setminus V \mid v_\delta(z) = 0\} = \varnothing$.

14. Q.E.D.

   Proof: Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}/2$. Let $v := v_\delta$. By steps 8 to 13, v is the desired map. □

12 Exercise sheet No. 12 - 11-07-2019

Let M be an oriented manifold, Σ_0, Σ_1 be oriented manifolds of dimension 2, and for $j = 0,1$ let $u_j\colon \Sigma_j \longrightarrow M$ be a smooth map. $(z_0, z_1) \in \Sigma_0 \times \Sigma_1$ is an **isolated intersection** of u_0, u_1 if $u_0(z_0) = u_1(z_1)$ and there exist U_0 a neighborhood of z_0 and U_1 a neighborhood of z_1 such that

- U_0, U_1 have compact closure,
- $\forall w_0 \in \overline{U_0}\colon \forall w_1 \in \overline{U_1}\colon (u_0(w_0) = u_1(w_1) \Longrightarrow w_0 = z_0, w_1 = z_1)$.

Let $(z_0, z_1) \in \Sigma_0 \times \Sigma_1$ be an isolated intersection of u_0, u_1. We define the **local intersection number** of u_0, u_1 at (z_0, z_1), denoted $\iota(u_0, u_1; z_0, z_1)$, as follows. Choose

- U_0, U_1 as before,
- $W \subset M$ compact, contractible,
- $v_0\colon U_0 \longrightarrow M$, $v_1\colon U_1 \longrightarrow M$

such that

- $u_0(z_0) = u_1(z_1) \in W$,
- $W \cap u_0(\partial U_0) = W \cap u_1(\partial U_1) = \varnothing$,
- $v_0 \pitchfork v_1$,
- $\forall j = 0,1\colon v_j|_{U_j \setminus u_j^{-1}(W)} = u_j|_{U_j \setminus u_j^{-1}(W)}$,
- $\forall j = 0,1\colon v_j(U_j \cap u_j^{-1}(W)) \subset W$.

Then, $\iota(u_0, u_1; z_0, z_1) \coloneqq v_0 \cdot v_1$.

The previous definition is well posed, i.e. $\iota(u_0, u_1; z_0, z_1)$ does not depend on the choice of date used to define it.

Exercise 12.1. Let (M, J) be an almost complex 4-dimensional manifold, (Σ, j) be a Riemann surface, and $u\colon \Sigma \longrightarrow M$ be a simple J-holomophic curve. Let $(z_0, z_1) \in \Sigma \times \Sigma \setminus \Delta$ be such that $u(z_0) = u(z_1)$. Then,

(i) (z_0, z_1) is an isolated self-intersection of u;

(ii) $\iota(u, u; z_0, z_1) \geq 1$;

(iii) $\iota(u, u; z_0, z_1) = 1 \Longleftrightarrow \operatorname{im} du(z_0) \oplus \operatorname{im} du(z_1) = T_{u(z_0)}M$.

Solution:

1. (i):

   Proof: By exercise 11.1 and u being simple.

2. For $i = 0,1$, there exist
   - C^2-coordinate charts $\varphi_i\colon (U_i, z_i) \longrightarrow (\mathbb{C}, 0)$,

- unitary matrices $L_i \in \mathbb{C}^{2\times 2}$,
- a C^1-coordinate chart $\Psi\colon (W_0, u(z_0)) \longrightarrow (\mathbb{C}^2, 0)$

such that

- U_0, U_1 are disjoint,
- $d\Psi(u(z_0))J(x_0) = J_0 d\Psi(x_0)$,
- $\forall i = 0,1\colon u_i := \Psi \circ u \circ \varphi_i^{-1}\colon \varphi_i(U_i) \longrightarrow \mathbb{C}^2$ is a polynomial in the variable z,
- $\forall i = 0,1\colon \exists k_i \in \mathbb{N}\colon \exists p_i\colon \mathbb{C} \longrightarrow \mathbb{C}$ a polynomial with $p_i(0) = 0\colon \forall z \in \varphi_i(U_i)\colon$
$$L_i u_i(z) = (z^{k_i}, z^{k_i} p_i(z)),$$
- $L_0^{-1}(\mathbb{C}\times\{0\}) = L_1^{-1}(\mathbb{C}\times\{0\}) \iff L_0 = L_1$.

Proof: By the Micallef-White theorem.

3. Case $k_0 = 1$ and $k_1 = 1$.

   Proof:

   3.1. Case $du_0(0)\cdot\mathbb{C} + du_1(0)\cdot\mathbb{C} = \mathbb{C}^2$.

   Proof: In the definition of $\iota(u, u; z_0, z_1)$ we can take $v_i := u_i \circ \varphi_i$ and $W = \{u(z_0)\}$, and compute $\iota(u, u; z_0, z_1) = 1$. Therefore (ii) and (iii) are true.

   3.2. Case $du_0(0)\cdot\mathbb{C} + du_1(0)\cdot\mathbb{C} \neq \mathbb{C}^2$.

   Proof:

   3.2.1. We may assume that
   1. $L_0 = L_1 = \mathrm{id}_{\mathbb{C}}$,
   2. $p_0 = 0$,
   3. $\exists q\colon \mathbb{C} \longrightarrow \mathbb{C}$ polynomial $\colon \exists k \geq 2\colon q(0) \neq 0$ and $\forall z \in \mathbb{C}\colon p_1(z) = z^k q(z)$.

   Therefore, $u_0(z) = (z, 0)$ and $u_1(z) = (z, z^k q(z))$.

   Proof:
$$\begin{aligned} &du_0(0)\cdot\mathbb{C} + du_1(0)\cdot\mathbb{C} \neq \mathbb{C}^2 \\ &\quad\Longrightarrow u_0'(0), u_1'(0) \text{ are colinear} \\ &\quad\Longrightarrow L_0^{-1}(\mathbb{C}\times\{0\}) = L_1^{-1}(\mathbb{C}\times\{0\}) \\ &\quad\Longrightarrow L_0 = L_1. \end{aligned}$$
   Therefore, 1 follows from composing u_i with L_0 and 2, 3 follow from composing again with $\Psi'(z, w) = (z, w - zp_0(z))$.

   3.2.2. Let
   - $O \subset \mathbb{C}$ be a neighborhood of 0 such that $O \subset \varphi_1(U_1)$ and $\forall z \in O\colon p_1(z) \neq 0$.
   - $\beta\colon \mathbb{C} \longrightarrow [0,1]$ be smooth such that $\operatorname{supp}\beta \subset O$ and $\exists V$ a neighborhood of 0 such that $V \subset O, \beta|_V = 1$,
   - for each $\varepsilon > 0$, $f_\varepsilon(z) := \prod_{j=0}^{k-1}(z + j\varepsilon)$,
   - for each $\varepsilon > 0$, $\hat{v}_1^\varepsilon(z) = (z, (z^k + \beta(z)(f_\varepsilon(z) - z^k))q(z))$.

   3.2.3. Let
   - $W' := \{u_0(z_0)\}$,
   - $v_0 := u|_{U_0} \in C^\infty(U_0, M)$,

- For each $\varepsilon > 0$, $\tilde{v}_1^\varepsilon := \psi^{-1} \circ v_1^\varepsilon \circ \varphi_1 \in C^1(U_1, M)$.

3.2.4. There exists an ε such that
- the data $W', v_0, v_1^\varepsilon =: v_1$ is as in the definition of local intersection number.
- $u_0 \cdot v_1^\varepsilon = k$.

3.2.5. (ii):

Proof:

$$\begin{aligned} \iota(u, u; z_0, z_1) &= v_0 \cdot v_1 && \text{[by def. of } \iota \text{ and step 3.2.4]} \\ &= u_0 \cdot v_1^\varepsilon && [v_0 = u|_{U_0} \text{ and homotopy invariance]} \\ &= k \\ &\geq 2. \end{aligned}$$

3.2.6. (iii):

Proof: By assumption from step 3.2 and by step 3.2.5, both sides of the equivalence are false, so the result is true.

3.2.7. Q.E.D.

3.3. Q.E.D.

4. Case $k_0 \neq 1$ or $k_1 \neq 1$.

Proof:

4.1. $\exists r > 0$:
- $U := B_r(0) \subset \varphi_0(U_0) \cap \varphi_1(U_1)$
- $\forall w_0, w_1 \in \overline{U} \setminus \{0\} \colon u_0(w_0) \neq u_1(w_1)$.

Proof: u is simple.

4.2. $\exists r' > 0$:
- $W := \overline{B_{r'}(0)} \subset \psi(W_0)$,
- $W \cap u_0(\partial U) = \varnothing$,
- $W \cap u_1(\partial U) = \varnothing$.

4.3. $\forall i = 0, 1 \colon \exists r_i > 0$:
- $V_i := \overline{B_{r_i}(0)}$,
- $u_i(V_i) \subset \operatorname{int}(W)$.

4.4. $\exists \varepsilon > 0$:
- $\inf_{w_0 \in U, w_1 \in U \setminus V_1} |u_0(w_0) - u_1(w_1)| > 3\varepsilon$,
- $\inf_{w_0 \in U \setminus V_0, w_1 \in U} |u_0(w_0) - u_1(w_1)| > 3\varepsilon$.

Proof: By step 4.1, $\forall w_0, w_1 \in \overline{U} \setminus \{0\} \colon u_0(w_0) \neq u_1(w_1)$. $0 \in V_0$, $0 \in V_1$.

4.5. There exist $v_0 \colon U \longrightarrow \mathbb{C}^n$, $v_1 \colon U \longrightarrow \mathbb{C}^n$ such that
- v_i is smooth,
- v_i is an immersion,
- $v_i|_{U \setminus u_i^{-1}(W)} = u|_{U \setminus u_i^{-1}(W)}$,
- $v_i(U_i \cap u_i^{-1}(W)) \subset W$,

- v_i is holomorphic in V,
- $\|u_i - v_i\|_{C^1} < \varepsilon$,
- $\#\{v_i^{-1}(0)\} = k_i$,
- $v_0 \pitchfork v_1$.

Proof: By exercise 11.2, we can find some v_0, v_1 satisfying all the properties except transversality. We may deform v_0, v_1 such that transversality is true as well.

4.6. The data $U'_0 := \varphi_i^{-1}(U)$, $U'_1 := \varphi_i^{-1}(U)$, $W' = \psi^{-1}(W)$, $v'_0 := \psi^{-1} \circ v_0 \circ \varphi_0$, $v'_1 := \psi^{-1} \circ v_1 \circ \varphi_1$ satisfies the properties in the definition of ι.

Proof: By steps 4.1, 4.2 and 4.5.

4.7. $v_0 \cdot v_1 \geq k_0 \cdot k_1$.

Proof:

4.7.1. $\forall w_0, w_1 \in U \colon v_0(w_0) = v_1(w_1) \Longrightarrow w_0 \in V_0, w_1 \in V_1$.

Proof: Since v_i is C_1-ε-close to u_i, and by the inequalities in step 4.4.

4.7.2. $\#\{(w_0, w_1) \in U \times U \mid v_0(w_0) = v_1(w_1)\} \geq k_0 k_1$.

Proof: v_i has k_i branches through the origin.

4.7.3. $\forall w_0, w_1 \in U \colon \Big(v_0(w_0) = v_1(w_1) \Longrightarrow \operatorname{sign}\Big\{ dv_0(w_0) \cdot \frac{\partial}{\partial x}, dv_0(w_0) \cdot \frac{\partial}{\partial y}, dv_1(w_1) \cdot \frac{\partial}{\partial x}, dv_1(w_1) \cdot \frac{\partial}{\partial y} \Big\} = 1 \Big)$.

Proof: $v_0 \pitchfork v_1$ and by step 4.7.1, w_0, w_1 are in V, and v_0, v_1 are holomorphic in V.

4.7.4. Q.E.D.

4.8. $\iota(u, u; z_0, z_1) > 1$.

Proof:

$$\begin{aligned} \iota(u, u; z_0, z_1) &= (\psi^{-1} \circ v_0 \circ \varphi_0) \cdot (\psi^{-1} \circ v_1 \circ \varphi_1) && \text{[step 4.6]} \\ &= v_0 \cdot v_1 && \text{[invariance under diffeo.]} \\ &\geq k_0 k_1 && \text{[step 4.7]} \\ &> 1 && \text{[assumption in step 4].} \end{aligned}$$

4.9. (ii):

Proof: By step 4.8.

4.10. (iii):

Proof: Since $k_0 \neq 1$ or $k_1 \neq 1$, u_0, u_1 are not transverse. Therefore, by step 4.8, both sides of the equivalence are false, so the equivalence is true.

4.11. Q.E.D.

5. Q.E.D.

Proof: By steps 3 and 4. □

Exercise 12.2. Let (M, J) be an almost complex 4 dimensional manifold. For $i = 0, 1$, let $u_i \colon \Sigma_i \longrightarrow M$ be a simple J-holomorphic curve and $A_i := [u_i] \in H_2(M; \mathbb{Z})$. Let $U_0 \subset \Sigma_0, U_1 \subset \Sigma_1$ be open subsets such that $u_0(U_0) \neq u_1(U_1)$. Then,

(i) $\delta(u_0, u_1) := \#\{(z_0, z_1) \in \Sigma_0 \times \Sigma_1 \mid u_0(z_0) = u_1(z_1) \leq A_0 \cdot A_1\}$.

(ii) $\delta(u_0, u_1) = A_0 \cdot A_1$ if and only if

$$\forall z_0, z_1 \in \Sigma \colon \Big(u_0(z_0) = u_1(z_1) \Longrightarrow \operatorname{im} du_0(z_0) + \operatorname{im} du_1(z_1) = T_{u(z_0)}M \Big).$$

Solution:

1. $u := u_0 \sqcup u_1 \colon \Sigma_0 \sqcup \Sigma_1 \longrightarrow M$ is simple.

   Proof: Since u_0, u_1 are simple and $u_0(U_0) \neq u_1(U_1)$.

2. $\forall (z_0, z_1) \in \Sigma \times \Sigma \setminus \Delta$ self intersection of u,
   - (z_0, z_1) is isolated,
   - $\iota(u, u; z_0, z_1) \geq 1$,
   - $\iota(u, u; z_0, z_1) = 1 \Longleftrightarrow \operatorname{im} du(z_0) + \operatorname{im} du(z_1) = T_{u(z_0)}M$.

   Proof: By exercise 12.1.

3. (i):

   Proof:

$$\begin{aligned} A_0 \cdot A_1 &= \sum_{u(z_0)=u(z_1)} \iota(u_0, u_1; z_0, z_1) \\ &\geq \sum_{u(z_0)=u(z_1)} 1 \\ &= \delta(u_0, u_1). \end{aligned}$$

4. (ii):

   Proof:

$$\begin{aligned} &\delta(u_0, u_1) = A_0 \cdot A_1 \\ &\quad \Longleftrightarrow \delta(u_0, u_1) = \sum_{u(z_0)=u(z_1)} \iota(u_0, u_1; z_0, z_1) \\ &\quad \Longleftrightarrow \forall (z_0, z_1) \in \Sigma_0 \times \Sigma_1 \text{ s.t. } u_0(z_0) = u_1(z_1) \colon \iota(u_0, u_1; z_0, z_1) = 1 \\ &\quad \Longleftrightarrow \forall (z_0, z_1) \in \Sigma_0 \times \Sigma_1 \text{ s.t. } u_0(z_0) = u_1(z_1) \colon \operatorname{im} du(z_0) + \operatorname{im} du(z_1) = T_{u(z_0)}M. \end{aligned}$$

5. Q.E.D.

   Proof: Steps 3 and 4. □

References

[AH19] C. Abbas and H. Hofer. *Holomorphic curves and global questions in contact geometry.* Number 4842 in Birkhauser advanced texts basler lehrbucher. Springer Berlin Heidelberg, New York, NY, 2019.

[FHS95] Andreas Floer, Helmut Hofer, and Dietmar Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J.*, 80(1):251–292, 10 1995.

[MS04] D. McDuff and D. Salamon. *J-holomorphic Curves and Symplectic Topology*, volume 52 of *Colloquium Publications.* American Mathematical Society, first edition, 2004.

[MS17] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, third edition, 2017.

[MW97] J.W. Milnor and D.W. Weaver. *Topology from the Differentiable Viewpoint.* Princeton Landmarks in Mathematics. Princeton University Press, 1997.

[RY00] B.P. Rynne and M.A. Youngson. *Linear Functional Analysis.* Springer Undergraduate Mathematics Series. Springer, 2000.

[Wen15] C. Wendl. Lectures on holomorphic curves in symplectic and contact geometry, May 2015.