Topology - solutions to exercises

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1 Exercise sheet No. 1

Exercise 1.1. Determine all possible topologies on the set of three elements $X = \{a, b, c\}$.

Solution. Here is the complete list. Every topology in the same group is homeomorphic to each other, and no topologies from different groups are homeomorphic:

(A) (Trivial Topology) $\mathcal{O} = \{X, \varnothing\}$.

(B) (Singles)

- (B1) $\mathcal{O} = \{X, \varnothing, \{a\}\}$.
- (B2) $\mathcal{O} = \{X, \varnothing, \{b\}\}$.
- (B3) $\mathcal{O} = \{X, \varnothing, \{c\}\}$.

(C) (Doubles)

- (C1) $\mathcal{O} = \{X, \varnothing, \{a, b\}\}$.
- (C2) $\mathcal{O} = \{X, \varnothing, \{a, c\}\}$.
- (C3) $\mathcal{O} = \{X, \varnothing, \{b, c\}\}$.

(D) (single-doubles non-disjoint)

- (D1) $\mathcal{O} = \{X, \varnothing, \{a\}, \{a, b\}\}$.
- (D2) $\mathcal{O} = \{X, \varnothing, \{a\}, \{a, c\}\}$.
- (D3) $\mathcal{O} = \{X, \varnothing, \{b\}, \{b, c\}\}$.
- (D4) $\mathcal{O} = \{X, \varnothing, \{b\}, \{a, b\}\}$.
- (D5) $\mathcal{O} = \{X, \varnothing, \{c\}, \{a, c\}\}$.
- (D6) $\mathcal{O} = \{X, \varnothing, \{c\}, \{b, c\}\}$.

(E) (single-doubles disjoint)

- (E1) $\mathcal{O} = \{X, \varnothing, \{a\}, \{b, c\}\}$.
- (E2) $\mathcal{O} = \{X, \varnothing, \{b\}, \{a, c\}\}$.
- (E3) $\mathcal{O} = \{X, \varnothing, \{c\}, \{a, b\}\}$.

(F) (single-single-doubles)

- (F1) $\mathcal{O} = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$.
- (F2) $\mathcal{O} = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$.
- (F3) $\mathcal{O} = \{X, \varnothing, \{b\}, \{c\}, \{b, c\}\}$.

(G) (single-double-doubles)

- (G1) $\mathcal{O} = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\}$.

(G2) $\mathcal{O} = \{X, \varnothing, \{b\}, \{a,b\}, \{b,c\}\}$.

(G3) $\mathcal{O} = \{X, \varnothing, \{c\}, \{a,c\}, \{b,c\}\}$.

(I) (single-single-double-doubles)

(I1) $\mathcal{O} = \{X, \varnothing, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$.

(I2) $\mathcal{O} = \{X, \varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$.

(I3) $\mathcal{O} = \{X, \varnothing, \{a\}, \{c\}, \{a,b\}, \{a,c\}\}$.

(I4) $\mathcal{O} = \{X, \varnothing, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$.

(I5) $\mathcal{O} = \{X, \varnothing, \{b\}, \{c\}, \{b,c\}, \{a,c\}\}$.

(I6) $\mathcal{O} = \{X, \varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$.

(J) (Power set) $\mathcal{O} = \mathcal{P}(X) = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$. □

Exercise 1.2. Assume that X is a set and d_1, d_2 are two distance functions for X. Assume that there exists a constant $c > 0$ such that for every pair of points $x, y \in X$ the following string of inequalities holds

$$\frac{1}{c} d_1(x,y) \leq d_2(x,y) \leq c d_1(x,y). \tag{1}$$

Show that $\mathcal{O}_{d_1} = \mathcal{O}_{d2}$.

Solution. Let $c' = \frac{1}{c}$. Then, the inequalities (1) imply that

$$\frac{1}{c'} d_2(x,y) \leq d_1(x,y) \leq c' d_2(x,y).$$

So, it suffices to show that $\mathcal{O}_{d_1} \subset \mathcal{O}_{d_2}$, because if we prove this, then this inclusion with the roles of d_1 and d_2 switched tells us that $\mathcal{O}_{d_2} \subset \mathcal{O}_{d_1}$. To prove $\mathcal{O}_{d_1} \subset \mathcal{O}_{d_2}$, it suffices to assume that $U \in \mathcal{O}_{d_1}$ and prove that $U \in \mathcal{O}_{d_2}$. By definition of $\mathcal{O}_{d_1}$, for each $x \in U$ there exists an $\epsilon_x > 0$ such that $B^1_{\epsilon_x}(x) = \{y \in X \mid d_1(x,y) < \epsilon_x\} \subset U$. We claim that $U = \bigcup_{x \in U} B^1_{\epsilon_x}(x)$. To show $(\subset)$, let $x \in U$. Then $x \in B^1_{\epsilon_x}(x) \subset \bigcup_{y \in U} B^1_{\epsilon_y}(x)$. To show $(\supset)$, let $x \in \bigcup_{y \in U} B^1_{\epsilon_y}(y)$. Then there exists a $y \in U$ such that $x \in B^1_{\epsilon_y}(y)$. Since $B^1_{\epsilon_y}(y) \subset U$, $x \in U$. So $U = \bigcup_{x \in U} B^1_{\epsilon_x}(x)$. To show that $U \in \mathcal{O}_{d_2}$, since $\mathcal{O}_{d_2}$ is a topology (therefore closed under arbitrary unions) it suffices to assume that $x \in U$ and prove that $B^1_{\epsilon_x}(x) \in \mathcal{O}_{d_2}$. To show this, by definition of $\mathcal{O}_{d_2}$ it suffices to assume that $y \in B^1_{\epsilon_x}(x)$ and prove that there exists an $\epsilon_y > 0$ such that $B^2_{\epsilon_y}(y) \subset B^1_{\epsilon_x}(x)$. Define $\epsilon'_y := \epsilon_x - d_1(x,y)$. Then since $y \in B^1_{\epsilon_x}(x)$, $\epsilon'_y > 0$. We claim that $B^1_{\epsilon'_y}(y) \subset B^1_{\epsilon_x}(x)$. To see this, it suffices to assume that $z \in B^1_{\epsilon'_y}(y)$ and prove that $d_1(x,z) < \epsilon_x$.

$$\begin{aligned} d_1(x,z) &\leq d_1(x,y) + d_1(y,z) && \text{[by the triangular inequality]} \\ &< d_1(x,y) + \epsilon'_y && [z \in B^1_{\epsilon'_y}(y)] \\ &= \epsilon_x && \text{[by definition of } \epsilon'_y\text{].} \end{aligned}$$

Define $\epsilon_y = \epsilon'_y / c$. Then, $\epsilon_y > 0$. We claim that $B^2_{\epsilon_y}(y) \subset B^1_{\epsilon'_y}(y)$. To see this, it suffices to assume that $z \in B^2_{\epsilon_y}(y)$ and prove that $d_1(z,y) < \epsilon'_y$.

$$d_1(z,y) \leq c d_2(z,y) \quad \text{[by hypothesis of the exercise]}$$

$$
\begin{aligned}
&< c\epsilon_y && [z \in B^2_{\epsilon_y}(y)] \\
&= \epsilon'_y && [\text{by definition of } \epsilon_y].
\end{aligned}
$$

Using the previous two set inclusions,

$$
\begin{aligned}
B^2_{\epsilon_y}(y) &\subset B^1_{\epsilon'_y}(y) \\
&\subset B^1_{\epsilon_x}(x).
\end{aligned}
$$

□

Exercise 1.3. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$ be a subset of the power set of X that satisfies

(i) If I is any non-empty index set and A_i is closed for every $i \in I$, then $\bigcap_{i \in I} A_i$ is also closed.

(ii) If I is any non-empty finite index set and A_i is closed for every $i \in I$, then $\bigcup_{i \in I} A_i$ is also closed.

(iii) $\varnothing, X \in \mathcal{A}$.

Define $\mathcal{O} = \{U \subset X \mid U^c \in \mathcal{A}\}$. Show that $\mathcal{O}$ is a topology on X. (Hint: use De Morgan's laws)

Solution. $\mathcal{O}$ is closed under arbitrary unions: it suffices to assume that I is a non-empty set, $\forall i \in I\colon U_i \in \mathcal{O}$, and prove that $\bigcup_{i \in I} U_i \in \mathcal{O}$. By definition of $\mathcal{O}$, $\forall i \in I\colon U_i^c \in \mathcal{A}$. Then,

$$
\begin{aligned}
\bigcup_{i \in I} U_i \in \mathcal{O} &\iff \Big(\bigcup_{i \in I} U_i\Big)^c \in \mathcal{A} && [\text{definition of } \mathcal{O}] \\
&\iff \bigcap_{i \in I} U_i^c \in \mathcal{A} && [\text{De Morgan's laws}] \\
&\iff \text{true} && [\mathcal{A} \text{ satisfies (i)}].
\end{aligned}
$$

$\mathcal{O}$ is closed under finite intersections: it suffices to assume that I is a finite non-empty set, $\forall i \in I\colon U_i \in \mathcal{O}$, and prove that $\bigcap_{i \in I} U_i \in \mathcal{O}$. By definition of $\mathcal{O}$, $\forall i \in I\colon U_i^c \in \mathcal{A}$. Then,

$$
\begin{aligned}
\bigcap_{i \in I} U_i \in \mathcal{O} &\iff \Big(\bigcap_{i \in I} U_i\Big)^c \in \mathcal{A} && [\text{definition of } \mathcal{O}] \\
&\iff \bigcup_{i \in I} U_i^c \in \mathcal{A} && [\text{De Morgan's laws}] \\
&\iff \text{true} && [\mathcal{A} \text{ satisfies (ii)}].
\end{aligned}
$$

$\varnothing \in \mathcal{O}$:

$$
\begin{aligned}
\varnothing \in \mathcal{O} &\iff \varnothing^c \in \mathcal{A} && [\text{by definition of } \mathcal{O}] \\
&\iff X \in \mathcal{A} && [\varnothing^c = X] \\
&\iff \text{true} && [\mathcal{A} \ \text{ satisfies (iii)}].
\end{aligned}
$$

$X \in \mathcal{O}$:

$$
\begin{aligned}
X \in \mathcal{O} &\iff X^c \in \mathcal{A} && [\text{by definition of } \mathcal{O}] \\
&\iff \varnothing \in \mathcal{A} && [X^c = \varnothing] \\
&\iff \text{true} && [\mathcal{A} \ \text{ satisfies (iii)}].
\end{aligned}
$$

□

Exercise 1.4. Consider the Euclidean distance function on $\mathbb{R}^n$, which for two vectors $x, y \in \mathbb{R}^n$ is defined by

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Show that the closure of the open ball is the closed ball. That is, for $x \in \mathbb{R}^n$ and $r > 0$ show that

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid d(x, y) \leq r\}.$$

Solution. Define $C_r(x) := \{y \in \mathbb{R}^n \mid d(x, y) \leq r\}$. We show that $\overline{B_r(x)} \subset C_r(x)$. For this, by definition of closure it suffices to show that $B_r(x) \subset C_r(x)$ and that $C_r(x)$ is closed in $\mathbb{R}^n$. Clearly $B_r(x) \subset C_r(x)$. To show that $C_r(x)$ is closed, it suffices to show that $\mathbb{R}^n \setminus C_r(x)$ is open. For this, by definition of the topology of $\mathbb{R}^n$ as a metric space it suffices to assume that $y \in \mathbb{R}^n \setminus C_r(x)$ and prove that there exists an $\epsilon > 0$ such that $B_\epsilon(y) \subset \mathbb{R}^n \setminus C_r(x)$. Define $\epsilon := d(x, y) - r$. $\epsilon > 0$, because

$$\begin{aligned} y \in \mathbb{R}^n \setminus C_r(x) &\iff y \notin C_r(x) \\ &\iff \neg(d(x, y) \leq r) \\ &\iff d(x, y) > r. \end{aligned}$$

We now show that $B_\epsilon(y) \subset \mathbb{R}^n \setminus C_r(x)$. For this, it suffices to assume that $z \in B_\epsilon(y)$ (i.e. $d(y, z) < \epsilon$) and prove that $z \neq C_r(x)$ (i.e. $d(x, z) > r$).

$$\begin{aligned} r &< d(x, y) - d(y, z) && [d(y, z) < \epsilon \text{ and definition of } \varepsilon] \\ &\leq d(x, z) && [\text{by the triangle inequality}]. \end{aligned}$$

We show that $C_r(x) \subset \overline{B_r(x)}$. By definition of closure, it suffices to assume that F is closed, $\overline{B_r(x)} \subset F$, and prove that $C_r(x) \subset F$. For this, it suffices to assume that $y \in C_r(x)$ and prove that $y \in F$. Assume by contradiction that $y \notin F$. Then, since $y \in \mathbb{R}^n \setminus F$ which is open, and by definition of open set of the metric topology, there exists $\epsilon' > 0$ such that $B_{\epsilon'}(y) \subset \mathbb{R}^n \setminus F$. Let $\epsilon = \min\{\epsilon', \|x - y\|\}$. Then, $B_\epsilon(y) \subset \mathbb{R}^n \setminus F$ and $\epsilon < \|x - y\|$. $B_\epsilon(y) \cap B_r(x) \subset (\mathbb{R}^n \setminus F) \cap F = \varnothing$. Define $t := \frac{\epsilon}{2\|x-y\|}$ and $z = y + (x - y)t$. We will derive a contradiction by showing that $z \in B_\epsilon(y) \cap B_r(x)$, which we have proven is empty. $z \in B_\epsilon(y)$:

$$\begin{aligned} d(y, z) &= d(y, y + (x - y)t) && [\text{by definition of } z] \\ &= \|-y + y + (x - y)t\| && [\text{by definition of } d] \\ &= \|t(x - y)\| \\ &= |t|\|x - y\| && [\|\cdot\| \text{ is a norm}] \\ &= \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

$z \in B_r(x)$:

$$d(x, z) = d(x, y + (x - y)t) \qquad [\text{definition of } z]$$

$$
\begin{aligned}
&= \|-x + y + xt - yt\| && [\text{definition of } d] \\
&= \|(t-1)(x-y)\| \\
&= |t-1|\|x-y\| && [\|\cdot\| \text{ is a norm}] \\
&= |\frac{\epsilon}{2} - \|x-y\|| && [\text{by definition of } t] \\
&= \|x-y\| - \frac{\epsilon}{2} && [\|x-y\| > \epsilon > \epsilon/2] \\
&\leq r - \epsilon/2 && [y \in C_r(x)] \\
&< r.
\end{aligned}
$$

So, $z \in B_\epsilon(y) \cap B_r(x) = \varnothing$, which is a contradiction. $\square$

2 Exercise sheet No. 2

Exercise 2.1. Let (X, d) be a metric space, $x \in X$ and $r > 0$. Show that

$$\overline{B_r(x)} \subset \{y \in X \mid d(x, y) \leq r\}.$$

Solution. In exercise 1.4 we showed that

$$\overline{B_r(x)} = \{y \in X \mid d(x, y) \leq r\}$$

in the case $X = \mathbb{R}^n$. The proof we gave for $(\supset)$ uses the fact that the space is $\mathbb{R}^n$ (namely the fact that it is a vector space), however the proof that we gave for $(\subset)$ only uses the fact that $\mathbb{R}^n$ is a metric space. So the same proof can be used here, with $\mathbb{R}^n$ replaced by X:

Define $C_r(x) := \{y \in X \mid d(x, y) \leq r\}$. We show that $\overline{B_r(x)} \subset C_r(x)$. For this, by definition of closure it suffices to show that $B_r(x) \subset C_r(x)$ and that $C_r(x)$ is closed in X. Clearly $B_r(x) \subset C_r(x)$. To show that $C_r(x)$ is closed, it suffices to show that $X \setminus C_r(x)$ is open. For this, by definition of the topology of X as a metric space it suffices to assume that $y \in X \setminus C_r(x)$ and prove that there exists an $\epsilon > 0$ such that $B_\epsilon(y) \subset X \setminus C_r(x)$. Define $\epsilon := d(x, y) - r$. $\epsilon > 0$, because

$$\begin{aligned} y \in X \setminus C_r(x) &\iff y \notin C_r(x) \\ &\iff \neg(d(x, y) \leq r) \\ &\iff d(x, y) > r. \end{aligned}$$

We now show that $B_\epsilon(y) \subset X \setminus C_r(x)$. For this, it suffices to assume that $z \in B_\epsilon(y)$ (i.e. $d(y, z) < \epsilon$) and prove that $z \neq C_r(x)$ (i.e. $d(x, z) > r$).

$$\begin{aligned} r &< d(x, y) - d(y, z) && [d(y, z) < \epsilon \text{ and definition of } \varepsilon] \\ &\leq d(x, z) && [\text{by the triangle inequality}]. \end{aligned}$$

□

Exercise 2.2. Find an example of a metric space (X, d) in which the inclusion from the previous exercise is strict. That is, for which there exists $x \in X$ and $r > 0$ such that

$$\overline{B_r(x)} \neq \{y \in X \mid d(x, y) \leq r\}.$$

Solution. Take any set X with more than one element, and consider the *discrete* metric d on X defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

This is indeed a metric, as is easy to check. Moreover, $B_1(x) = \overline{B_1(x)} = \{x\}$ for every x, but

$$\{y \in X \mid d(x, y) \leq 1\} = X.$$

□

Exercise 2.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Show that a map $f\colon X \to Y$ is continuous if and only if the preimage under f of every closed set is closed.

Solution. Assume that f is continuous, and let $A \subset Y$ be closed. Then

$$f^{-1}(A)^c = f^{-1}(A^c)$$

is open, which implies that $f^{-1}(A)$ is closed. The other direction is analogous. $\square$

Exercise 2.4. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f\colon (X, \mathcal{O}_{d_X}) \to (Y, \mathcal{O}_{d_Y})$ be a map. Show that f is continuous if and only if for every $\epsilon > 0$ and for every $x \in X$ there exists a $\delta = \delta(x, \epsilon) > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$.

Solution. ($\Longrightarrow$): It suffices to assume that $\epsilon > 0$, $x \in X$ and show that there exists a $\delta = \delta(x, \epsilon) > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$. By definition of metric space topology on Y, the set $B_\epsilon(f(x))$ is open in Y. Since f is continuous, $f^{-1}(B_\epsilon(f(x)))$ is open in X. Also $f(x) \in B_\epsilon(f(x))$. Therefore, by definition of the metric space topology on X, there exists a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. We claim that δ is as desired. To show this, it suffices to assume that $y \in X$ and prove that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$.

$$\begin{aligned}
d_X(x, y) < \delta &\iff y \in B_\delta(x) && \text{[definition of ball]}\\
&\implies y \in f^{-1}(B_\epsilon(f(x))) && \text{[by our choice of } \delta,\ B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))]\\
&\iff f(y) \in B_\epsilon(f(x)) && \text{[definition of preimage]}\\
&\iff d_Y(f(x), f(y)) < \epsilon && \text{[definition of ball].}
\end{aligned}$$

($\Longleftarrow$): By definition of continuous function, it suffices to assume that $U \subset Y$ is open and prove that $f^{-1}(U) \subset X$ is open. For this, by definition of open set in the metric space topology, it suffices to assume that $x \in f^{-1}(U)$ and prove that there exists a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(U)$. Since $f(x) \in U$, U is open, and by definition of metric space topology, there exists an $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$. By hypothesis, there exists a $\delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$ for all $y \in X$. We claim that $B_\delta(x) \subset f^{-1}(U)$. Since

$$\begin{aligned}
y \in B_\delta(x) &\iff d_X(x, y) < \delta && \text{[definition of ball]}\\
&\implies d_Y(f(x), f(y)) < \epsilon && \text{[by our choice of } \delta]\\
&\iff f(y) \in B_\epsilon(f(x)) && \text{[definition of ball]}\\
&\iff y \in f^{-1}(B_\epsilon(f(x))) && \text{[definition of preimage],}
\end{aligned}$$

$B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. Then, since $B_\epsilon(f(x)) \subset U$,

$$\begin{aligned}
B_\delta(x) &\subset f^{-1}(B_\epsilon(f(x)))\\
&\subset f^{-1}(U).
\end{aligned}$$

$\square$

3 Exercise sheet No. 3

Exercise 3.1 (Subset topology)**.** Let $A \subset (X, \mathcal{O}_X)$ be any subset. Show that $(A, \mathcal{O}_X|_A)$ is a topological space, and that the natural inclusion $\iota\colon A \hookrightarrow X$ is continuous.

Solution. We have $\varnothing = A \cap \varnothing$, $A = A \cap X$, and so $A, \varnothing \in \mathcal{O}_X|_A$. Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{O}_X|_A$, and write $U_i = A \cap V_i$ with $V_i \in \mathcal{O}_X$. Then

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} (A \cap V_i) = A \cap \left(\bigcup_{i\in I} V_i\right),$$

which is an element of $\mathcal{O}_X|_A$ since $\bigcup_{i\in I} V_i \in \mathcal{O}_X$. Similarly, if I is finite, then

$$\bigcap_{i\in I} U_i = \bigcap_{i\in I} (A \cap V_i) = A \cap \left(\bigcap_{i\in I} V_i\right),$$

which again lies in $\mathcal{O}_X|_A$ since $\bigcap_{i\in I} V_i \in \mathcal{O}_X$. This shows that $\mathcal{O}_X|_A$ is a topology.

To show that ι is continuous, take any open set $V \subset X$. Then $\iota^{-1}(V) = A \cap V$, which is an element of $\mathcal{O}_X|_A$ by definition. $\square$

Exercise 3.2 (Quotient topology)**.** Let $(X, \mathcal{O}_X)$ be a topological space and $\sim$ an equivalence relation on X. Show that $(X/\sim, \mathcal{O}_{X/\sim})$ is a topological space and the natural projection $p : X \to X/\sim$ is continuous.

Solution. The proof is completely analogous to that of the previous exercise.

Denote $Y = X/\sim$. We have $\varnothing = p^{-1}(\varnothing)$, $X = p^{-1}(Y)$, and so $Y, \varnothing \in \mathcal{O}_Y$. Let $\{U_i\}_{i\in I}$ be a family of elements in $\mathcal{O}_Y$. Then

$$p^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} p^{-1}(U_i),$$

which is an element of $\mathcal{O}_X$ since $p^{-1}(U_i) \in \mathcal{O}_X$ for each $i \in I$ and $\mathcal{O}_X$ is a topology. Similarly, if I is finite, then

$$p^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} p^{-1}(U_i),$$

which again lies in $\mathcal{O}_X$ for similar reasons. This shows that $\mathcal{O}_Y$ is a topology.

The fact that p is continuous follows directly from the definition of the quotient topology $\mathcal{O}_Y$. $\square$

Exercise 3.3. Show that $(\mathbb{R}/\mathbb{Z}, \mathcal{O}_{\mathbb{R}/\mathbb{Z}})$ is compact and that $(S^1, \mathcal{O}_{\mathbb{R}^2|S^1})$ is Hausdorff.

Solution. We show that $(\mathbb{R}/\mathbb{Z}, \mathcal{O}_{\mathbb{R}/\mathbb{Z}})$ is compact. For this, consider $f\colon [0,1] \longrightarrow \mathbb{R}/\mathbb{Z}$ given by $f(x) = [x]$. By Lemma 3.8, it suffices to show that $[0,1]$ is compact and that f is continuous and surjective (i.e. it follows that $f([0,1]) = \mathbb{R}/\mathbb{Z}$ is compact).

$[0,1]$ is compact: It suffices to assume that $\mathcal{A} = \{U_i\}_{i\in I}$ is an open covering of $[0,1]$ and to prove that there is a finite subcover of $\mathcal{A}$. Define

$$C = \{y \in (0,1] \mid [0,y] \text{ can be covered by finitely many elements of } \mathcal{A}\}.$$

It suffices to show that $C \neq \varnothing$, which implies that C has a supremum c, and that $c \in C$. We start then by showing that $C \neq \varnothing$. Choose $i \in I$ such that $0 \in U_i$. Since U_i is open, there exists an $x > 0$ such that $[0, x) \in U_i$. Then, $x/2 \in C$ and $C \neq \varnothing$. Let $c = \sup C$. Assume by contradiction that $c \notin C$. $c \in [0, 1]$, so we can choose $i \in I$ such that $c \in U_i$. Since U_i is open, there exists a $d \in [0, c)$ such that $(d, c] \subset U_i$. There exists a $z \in C$ such that that $z \in (d, c)$, since

$$
\begin{aligned}
\neg(\exists z \in C\colon z \in (d,c)) &\iff \forall z \in C\colon z \notin (d,c) \\
&\iff \forall z \in C\colon z \leq d && [z \in C \implies z \leq c, \\
& && \ z \in C \wedge c \notin C \implies z \neq c] \\
&\implies c \neq \sup C && [d \text{ is a smaller upper bound than } c] \\
&\iff \text{false}.
\end{aligned}
$$

$z \in C$ implies that there exist $i_1, \ldots, i_n \in I$ such that $[0, z] \subset \bigcup_{j=1}^n U_{i_j}$. Also, $[z, c] \subset (d, c] \subset U_i$. So, $[0, c] = [0, z] \cup [z, c] \subset \left(\bigcup_{j=1}^n U_{i_j}\right) \cup U_i$, which means that $[0, c]$ can be covered by a finitely many elements of $\mathcal{A}$, so $c \in C$ which contradicts $c \notin C$. So $c \in C$.

f is continuous: consider the following commutative diagram:

$$
\begin{tikzcd}
{[0,1]} \arrow[r, "\iota"] \arrow[dr, "f"'] & \mathbb{R} \arrow[d, "p"] \\
& \mathbb{R}/\mathbb{Z}
\end{tikzcd}
.
$$

By the previous two exercises, p and ι are continuous, and $f = p\iota$.

f is surjective: it suffices to assume that $[x] \in \mathbb{R}/\mathbb{Z}$ and to prove that there exists a $y \in [0, 1]$ such that $f(y) = [x]$. Define $y = x - \lfloor x \rfloor$. Then, $y \in [0, 1]$ and $f(y) = [x - \lfloor x \rfloor] = [x]$.

We show that $(S^1, \mathcal{O}_{\mathbb{R}^2|S^1})$ is Hausdorff. We prove the more general fact that if $(X, \mathcal{O}_X)$ is a Hausdorff topological space and $Y \subset X$, then $(Y, \mathcal{O}_{X|Y})$ is Hausdorff. For this, it suffices to assume that $x, y \in Y$ and to prove that there exist $U, V \subset Y$ open in Y such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. Since X is Hausdorff, there exist $U', V' \subset X$ open in X such that $x \in U'$, $y \in V'$, and $U' \cap V' = \varnothing$. Define $U = Y \cap U'$ and $V = Y \cap V'$. By definition of subspace topology, U, V are open in Y. Since $x, y \in Y$, $x \in U$ and $y \in V$. And $U \cap V = \varnothing$, since $U' \cap V' = \varnothing$. □

Exercise 3.4. Show that $(\mathbb{R}/\mathbb{Z}, \mathcal{O}_{\mathbb{R}/\mathbb{Z}})$ is homeomorphic to $(S^1, \mathcal{O}_{\mathbb{R}^2|S^1})$.

Solution. By theorem 3.7, it suffices to show that there exists a map $f\colon \mathbb{R}/\mathbb{Z} \longrightarrow S^1$ which is continuous and bijective. Define f by $f([t]) = (\cos(2\pi t), \sin(2\pi t))$ and define $\tilde{f}\colon \mathbb{R} \longrightarrow S^1$ by $\tilde{f}(t) = (\cos(2\pi t), \sin(2\pi t))$. Then, the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & S^1 \\
\mathbb{R}/\mathbb{Z} \arrow[ur, "f"']
\end{tikzcd}
.
$$

f is well defined: it suffices to assume that $t, t' \in \mathbb{R}$, $z \in \mathbb{Z}$, $t' = t + z$ and to prove that $(\cos(2\pi t'), \sin(2\pi t')) = (\cos(2\pi t), \sin(2\pi t))$. This is true:

$$\begin{aligned}
(\cos(2\pi t'), \sin(2\pi t')) &= (\cos(2\pi t + 2\pi z), \sin(2\pi t + 2\pi z)) \\
&= (\cos(2\pi t), \sin(2\pi t)) && [z \in \mathbb{Z}].
\end{aligned}$$

f is injective: it suffices to assume that $t, t' \in \mathbb{R}$ are such that $(\cos(2\pi t'), \sin(2\pi t')) = (\cos(2\pi t), \sin(2\pi t))$ and to prove that there exists a $z \in \mathbb{Z}$ such that $t' = t + z$.

$$\begin{aligned}
(\cos(2\pi t'), \sin(2\pi t')) = (\cos(2\pi t), \sin(2\pi t)) &\implies e^{2\pi i t} = e^{2\pi i t'} \\
&\implies e^{2\pi i (t - t')} = 1 \\
&\implies t - t' \in \mathbb{Z}.
\end{aligned}$$

f is surjective: for each $p \in S^1$ there exists a $t \in \mathbb{R}$ such that $p = (\cos(2\pi t), \sin(2\pi t))$.

f is continuous: it suffices to assume that $U \subset S^1$ is open and to prove that $f^{-1}(U)$ is open.

$$\begin{aligned}
f^{-1}(U) \text{ is open} &\iff p^{-1} f^{-1}(U) \text{ is open} && [\text{definition of quotient topology}] \\
&\iff (fp)^{-1}(U) \text{ is open} \\
&\iff \tilde{f}^{-1}(U) \text{ is open} && [\text{commutative diagram}] \\
&\iff \text{true} && [\tilde{f} \text{ is continuous}]. \qquad \square
\end{aligned}$$

4 Exercise sheet No. 4

Exercise 4.1. Consider a non-empty product space $X = \prod_{\lambda\in\Lambda} X_\lambda$. Show that X is Hausdorff if and only if X_λ is Hausdorff for every $\lambda \in \Lambda$.

Solution. ($\Longleftarrow$): Assume that X_λ is Hausdorff for every $\lambda \in \Lambda$, and let $x = (x_\lambda)_{\lambda\in\Lambda}$, $y = (y_\lambda)_{\lambda\in\Lambda}$ be distinct points in X. Let $\lambda_0 \in \Lambda$ be such that $x_{\lambda_0} \neq y_{\lambda_0}$. Since X_{λ_0} is Hausdorff, there exist open sets $U_{\lambda_0}, V_{\lambda_0}$ in X_{λ_0} which respectively contain x_{λ_0} and y_{λ_0}, and which are disjoint. Define $U_\lambda := V_\lambda := X_\lambda$ for $\lambda \neq \lambda_0$, and let

$$U := \prod_{\lambda\in\Lambda} U_\lambda,\ V := \prod_{\lambda\in\Lambda} V_\lambda$$

Then by definition U and V are elements in the natural basis for the product topology in X, containing respectively x and y, and which are disjoint by construction. Then X is Hausdorff.

($\Longrightarrow$): For the other direction, assume that X is Hausdorff. Fix $\lambda_0 \in \Lambda$, and consider $x_{\lambda_0} \neq y_{\lambda_0}$ distinct points in X_{λ_0}. Choose an arbitrary point $x_\lambda = y_\lambda \in X_\lambda$ for $\lambda \neq \lambda_0$ (here we use that $X_\lambda \neq \varnothing$ for every λ), and consider $x = (x_\lambda)_{\lambda\in\Lambda}$, $y = (y_\lambda)_{\lambda\in\Lambda} \in X$. Then $x \neq y$ by construction, and since X is Hausdorff, we may find open sets U and V, satisfying $x \in U$, $y \in V$, $U \cap V = \varnothing$. Since U and V are the union of elements in the natural basis for the product topology, we may assume without loss of generality that they are themselves elements in that basis, i.e. they are of the form $U = \prod_{\lambda\in\Lambda} U_\lambda$, $V = \prod_{\lambda\in\Lambda} V_\lambda$ with $U_\lambda = X_\lambda$ except for finitely many λ, and similarly for V_λ. If $z_{\lambda_0} \in U_{\lambda_0} \cap V_{\lambda_0}$, then we may define $z = (z_\lambda)_{\lambda\in\Lambda}$, where $z_\lambda = x_\lambda = y_\lambda$ for $\lambda \neq \lambda_0$. By construction $z \in U \cap V$, which is absurd. It follows that $U_{\lambda_0} \cap V_{\lambda_0} = \varnothing$, and so X_{λ_0} is Hausdorff. $\square$

Exercise 4.2. Show that the canonical projection $p_\lambda : X = \prod_{\mu\in\Lambda} X_\mu \to X_\lambda$ is continuous.

Solution. Let $U_\lambda \subset X_\lambda$ be an open set. Then

$$p_\lambda^{-1}(U_\lambda) = \prod_{\mu\in\Lambda} U_\mu,$$

where $U_\mu = X_\mu$ for $\mu \neq \lambda$. This is an element in the natural basis for the product topology, and therefore p_λ is continuous. $\square$

Exercise 4.3. Show that a topological space $(X, \mathcal{O})$ is Hausdorff if and only if there does not exist a filter of X that converges to two different points.

Solution. ($\Longrightarrow$): It suffices to assume that $\mathcal{F}$ is a filter on X, $x, y \in X$, $\mathcal{F}$ converges to x and $\mathcal{F}$ converges to y and to prove that $x = y$. Assume by contradiction that $x \neq y$. Then, since X is Hausdorff, there exist $U, V \subset X$ open such that $x \in U, y \in V$ and $U \cap V = \varnothing$. By definition of $\mathcal{U}_x$, $U \in \mathcal{U}_x$. Since $\mathcal{F}$ converges to x, $\mathcal{U}_x \subset \mathcal{F}$. So $U \in \mathcal{F}$. Analogously, $V \in \mathcal{F}$. Since

$$\begin{aligned} \varnothing &\notin \mathcal{F} && [\mathcal{F} \text{ is a filter}] \\ &\ni U \cap V && [U, V \in \mathcal{F} \text{ and def. of filter}] \end{aligned}$$

$$= \varnothing,$$

we obtain a contradiction.

($\Longleftarrow$): Assume by contradiction that X is not Hausdorff. We derive a contradiction by showing that there exists a filter $\mathcal{F}$ on X and points $x, y \in X$ such that $\mathcal{F}$ converges to x and $\mathcal{F}$ converges to y. Since X is not Hausdorff, there exist $x, y \in X$ such that if U is a neighbourhood of x and if V is a neighbourhood of y, then $U \cap V \neq \varnothing$. Define

$$\begin{aligned}\mathcal{F} = \{W \in \mathcal{P}(X) \mid \exists U \text{ a neighbourhood of } x\colon \\ \exists V \text{ a neighbourhood of } y\colon U \cap V \subset W\}.\end{aligned}$$

We show that $\mathcal{F}$ is a filter.

$\mathcal{F} \neq \varnothing$: because $X \in \mathcal{F}$.

$A \in \mathcal{F}$ and $A \subset B \Longrightarrow B \in \mathcal{F}$: Since $A \in \mathcal{F}$, there exists U a neighbourhood of x and there exists V a neighbourhood of y such that $U \cap V \subset A$. Then, $U \cap V \subset A \subset B$, so $B \in \mathcal{F}$.

$A \in \mathcal{F}$ and $B \in \mathcal{F} \Longrightarrow A \cap B \in \mathcal{F}$: Since $A \in \mathcal{F}$, there exists U a neighbourhood of x and there exists V a neighbourhood of y such that $U \cap V \subset A$. Since $B \in \mathcal{F}$, there exists U' a neighbourhood of x and there exists V' a neighbourhood of y such that $U' \cap V' \subset B$. Define $U'' = U \cap U'$ and $V'' = V \cap V'$. Then, $x \in U''$, $y \in V''$ and $A \cap B \supset (U \cap V) \cap (U' \cap V') = (U \cap U') \cap (V \cap V') = U'' \cap V''$, so $A \cap B \in \mathcal{F}$.

$\varnothing \notin \mathcal{F}$: Assume by contradiction $\varnothing \in \mathcal{F}$. Then there exist U a neighbourhood of x in X and V a neighbourhood of y in X such that $U \cap V \subset \varnothing$. By the property of x and y above (which comes from the fact that X is not Hausdorff), $U \cap V \neq \varnothing$. Contradiction.

So, we conclude that $\mathcal{F}$ is a filter.

We now show that $\mathcal{F}$ converges to x. It suffices to assume that $A \in \mathcal{U}_x$ and to prove that $A \in \mathcal{F}$. Since $A \in \mathcal{U}_x$, there exists a U open such that $x \in U \subset A$. Define $V \coloneqq X$, which is a neighbourhood of y. Then, $U \cap V = U \cap X = U \subset A$, so $A \in \mathcal{F}$. Analogously, $\mathcal{F}$ converges to y. This is a contradiction, because $x \neq y$ and filters on X have unique limits. $\square$

Exercise 4.4. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product of topological spaces and $\mathcal{F}$ be a filter on X. Show that $\mathcal{F}$ converges to $x = \{x_\lambda\}_{\lambda \in \Lambda}$ if and only if $p_\lambda(\mathcal{F}) \coloneqq \{p_\lambda(A) \mid A \in \mathcal{F}\}$ converges to x_λ for all $\lambda \in \Lambda$.

Solution. We start by showing that for all $\lambda \in \Lambda$, $p_\lambda(\mathcal{F})$ is a filter, so that we can talk about it converging to a point.

$p_\lambda(\mathcal{F}) \neq \varnothing$: Since $\mathcal{F}$ is a filter it is nonempty, so there exists $A \in \mathcal{F}$. Then, $p_\lambda(A) \in p_\lambda(\mathcal{F})$.

$A \in p_\lambda(\mathcal{F})$ and $A \subset B \Longrightarrow B \in p_\lambda(\mathcal{F})$: Since $A \in p_\lambda(\mathcal{F})$, there exists $A' \in \mathcal{F}$ such that $A = p_\lambda(A')$. Define $B' = p_\lambda^{-1}(B)$. Then, since p_λ is surjective, $B = p_\lambda(B')$. Since

$$\begin{aligned} A' &\subset p_\lambda^{-1}(p_\lambda(A')) && \text{[fact about preimages]} \\ &= p_\lambda^{-1}(A) && [A = p_\lambda(A')] \\ &\subset p_\lambda^{-1}(B) && [A \subset B] \\ &= B' && [\text{def. } B'], \end{aligned}$$

$A' \subset B'$. $\mathcal{F}$ is a filter and $A' \in \mathcal{F}$, so $B' \in \mathcal{F}$. Since $B = p_\lambda(B')$, $B \in p_\lambda(\mathcal{F})$.

$A \in p_\lambda(\mathcal{F})$ and $B \in p_\lambda(\mathcal{F}) \implies A \cap B \in p_\lambda(\mathcal{F})$: Since $A, B \in p_\lambda(\mathcal{F})$, there exist $A', B' \in \mathcal{F}$ such that $p_\lambda(A') = A$ and $p_\lambda(B') = B$. Since $\mathcal{F}$ is a filter $A' \cap B' \in \mathcal{F}$. Therefore $p_\lambda(A' \cap B') \in p_\lambda(\mathcal{F})$. Since $p_\lambda(A' \cap B') \subset p_\lambda(A') \cap p_\lambda(B') = A \cap B$, and using the previous property of filters that we just proved, $A \cap B \in p_\lambda(\mathcal{F})$.

$\varnothing \notin p_\lambda(\mathcal{F})$: Assume by contradiction $\varnothing \in p_\lambda(\mathcal{F})$. There exists $A \in \mathcal{F}$ such that $\varnothing = p_\lambda(A)$. Since $A \in \mathcal{F}$, $A \neq \varnothing$. Therefore $p_\lambda(A) \neq \varnothing$ which is a contradiction.

($\Longrightarrow$): It suffices to assume that $\lambda \in \Lambda$, that $A \in \mathcal{U}_{x_\lambda}$ (i.e. $\exists U \subset X_\lambda$ open such that $x_\lambda \in U \subset A$) and to prove that $A \in p_\lambda(\mathcal{F})$ (i.e. $\exists A' \in \mathcal{F} \colon A = p_\lambda(A')$). Define $U' = p_\lambda^{-1}(U)$ and $A' = p_\lambda^{-1}(A)$. Since

$$\begin{aligned} x_\lambda \in U &\Longrightarrow p_\lambda(x) \in U \\ &\Longrightarrow x \in p_\lambda^{-1}(U) \\ &\Longrightarrow x \in U', \end{aligned}$$

and

$$\begin{aligned} U \subset A &\Longrightarrow p_\lambda^{-1}(U) \subset p_\lambda^{-1}(A) \\ &\Longrightarrow U \subset A, \end{aligned}$$

then $A' \in \mathcal{U}_x$. Therefore $A' \in \mathcal{F}$. Since p_λ is surjective, $p_\lambda(A') = A$.

($\Longleftarrow$): It suffices to assume that $A \in \mathcal{U}_x$ and to prove that $A \in \mathcal{F}$. Since $A \in \mathcal{U}_x$, there exists $U \subset X$ open such that $x \in U \subset A$. By definition of the product topology, there exists a family $\{U_\lambda\}_{\lambda \in \Lambda}$, where $U_\lambda \subset X_\lambda$ is open for each λ, such that

$$x \in \prod_{\lambda \in \Lambda} U_\lambda \subset U$$

and there exist $\lambda_1, \ldots, \lambda_n$ such that for all $\lambda \in \Lambda \setminus \{\lambda_1, \ldots, \lambda_n\}$ we have that $U_\lambda = X_\lambda$. Then, $\prod_{\lambda \in \Lambda} U_\lambda = \bigcap_{i=1}^n p_{\lambda_i}^{-1}(U_{\lambda_i})$, because

$$\begin{aligned} y \in \prod_{\lambda \in \Lambda} U_\lambda &\Longleftrightarrow \forall \lambda \in \Lambda \colon y_\lambda \in U_\lambda \\ &\Longleftrightarrow \forall i = 1, \ldots, n \colon y_{\lambda_i} \in U_{\lambda_i} \\ &\Longleftrightarrow \forall i = 1, \ldots, n \colon p_{\lambda_i}(y) \in U_{\lambda_i} \\ &\Longleftrightarrow \forall i = 1, \ldots, n \colon y \in p_{\lambda_i}^{-1}(U_{\lambda_i}) \\ &\Longleftrightarrow y \in \bigcap_{i=1}^n p_{\lambda_i}^{-1}(U_{\lambda_i}). \end{aligned}$$

By definition of $\mathcal{U}_{x_\lambda}$, $U_\lambda \in \mathcal{U}_{x_\lambda} \subset p_\lambda(\mathcal{F})$ for every $\lambda \in \Lambda$. By definition of $p_\lambda(\mathcal{F})$, for every $\lambda \in \Lambda$ there exists an $A_\lambda \in \mathcal{F}$ such that $U_\lambda = p_\lambda(A_\lambda)$. By definition of filter, $\bigcap_{i=1}^n A_{\lambda_i} \in \mathcal{F}$. Since

$$\begin{aligned} \bigcap_{i=1}^n A_{\lambda_i} &\subset \bigcap_{i=1}^n p_{\lambda_i}^{-1}(U_{\lambda_i}) \quad [A_\lambda \subset p_\lambda^{-1}(p_\lambda(A_\lambda)) = p_\lambda^{-1}(U_\lambda)] \\ &= \prod_{\lambda \in \Lambda} U_\lambda \\ &\subset U \\ &\subset A \end{aligned}$$

and by definition of filter, $A \in \mathcal{F}$. $\square$

Exercise 4.5. Let X be a set and $x \in X$. Show that

$$\mathcal{F}_x = \{A \in \mathcal{P}(X) \mid x \in A\}$$

is an ultrafilter on X.

Solution. We show that $\mathcal{F}_x$ is a filter. $\mathcal{F}_x \neq \varnothing$: Since $x \in X$, $X \in \mathcal{F}_x$.

$A \in \mathcal{F}_x$ and $A \subset B \Longrightarrow B \in \mathcal{F}_x$: $A \in \mathcal{F}_x \Longrightarrow x \in A \Longrightarrow x \in B \Longrightarrow B \in \mathcal{F}_x$.

$A \in \mathcal{F}_x$ and $B \in \mathcal{F}_x \Longrightarrow A \cap B \in \mathcal{F}_x$: $A \in \mathcal{F}_x \wedge B \in \mathcal{F}_x \Longleftrightarrow x \in A \wedge x \in B \Longleftrightarrow x \in A \cap B \Longleftrightarrow A \cap B \in \mathcal{F}_x$.

$\varnothing \notin \mathcal{F}_x$: $\varnothing \in \mathcal{F}_x \Longrightarrow x \in \varnothing \Longleftrightarrow$ false.

So, $\mathcal{F}_x$ is a filter.

We show that $\mathcal{F}_x$ is an ultrafilter. It suffices to assume that $\mathcal{F}$ is a filter, $\mathcal{F}$ is finer than $\mathcal{F}_x$, and to prove that $\mathcal{F} = \mathcal{F}_x$. Since $\mathcal{F}$ is finer than $\mathcal{F}_x$, $\mathcal{F}_x \subset \mathcal{F}$. It remains to show the opposite inclusion. For this, it suffices to assume that $A \in \mathcal{F}$ and to prove that $x \in A$. Assume by contradiction that $x \notin A$. Then, $X \setminus A \in \mathcal{F}$, because

$$\begin{aligned} x \notin A &\Longleftrightarrow x \in X \setminus A \\ &\Longleftrightarrow X \setminus A \in \mathcal{F}_x \\ &\Longrightarrow X \setminus A \in \mathcal{F} \qquad [\mathcal{F}_x \subset \mathcal{F}]. \end{aligned}$$

Now $\varnothing = X \cap (X \setminus A) \in \mathcal{F}$ because $X, X \setminus A \in \mathcal{F}$. But since $\mathcal{F}$ is a filter $\varnothing \notin \mathcal{F}$. Contradiction. $\square$

5 Exercise sheet No. 5

Let Λ be a nonempty set. For each $\lambda \in \Lambda$, let X_λ be a nonempty set. Define

$$\mathcal{X} = \left\{ (L, z) \;\middle|\; L \subseteq \Lambda, z \in \prod_{\lambda \in L} X_\lambda \right\}.$$

Exercise 5.1. Show that $\mathcal{X}$ is nonempty.

Solution. Since Λ is nonempty, we can choose $\lambda \in \Lambda$. Since X_λ is nonempty, we can choose $z \in X_\lambda$. Define $L = \{\lambda\}$ (L is a set with one element). Then, $(L, z) \in \mathcal{X}$, because $L = \{\lambda\} \subseteq \Lambda$ and

$$\begin{aligned} \prod_{\lambda' \in L} X_{\lambda'} &= \prod_{\lambda' \in \{\lambda\}} X_{\lambda'} \\ &= X_\lambda \\ &\ni z. \end{aligned}$$

□

For $L_0 \subseteq L_1 \subseteq \Lambda$, define the canonical projection $p_{L_1,L_0} \colon \prod_{\lambda \in L_1} X_\lambda \longrightarrow \prod_{\lambda \in L_0} X_\lambda$ by $p_{L_1,L_0}((x_\lambda)_{\lambda \in L_1}) = (x_\lambda)_{\lambda \in L_0}$.

Define a binary relation on $\mathcal{X}$ by declaring $(L_1, z_1) \geq (L_0, z_0)$ if and only if $L_1 \supseteq L_0$ and $z_0 = p_{L_1,L_0} z_1$.

Exercise 5.2. Show that $\geq$ is a partial order.

Solution. Step 1: we show that if $L \subseteq \Lambda$, then $p_{L,L} = \mathrm{id} \colon \prod_{\lambda \in L} X_\lambda \longrightarrow \prod_{\lambda \in L} X_\lambda$. This is because if $(x_\lambda)_{\lambda \in L} \in \prod_{\lambda \in L} X_\lambda$, then by definition of the canonical projection $p_{L,L}((x_\lambda)_{\lambda \in L}) = (x_\lambda)_{\lambda \in L}$.

Step 2: we show that if $L_0 \subseteq L_1 \subseteq L_2 \subseteq \Lambda$, then $p_{L_1,L_0} p_{L_2,L_1} = p_{L_2,L_0}$. For this, notice that if $(x_\lambda)_{\lambda \in L_2}$, then

$$\begin{aligned} p_{L_1,L_0} p_{L_2,L_1}((x_\lambda)_{\lambda \in L_2}) &= p_{L_1,L_0}((x_\lambda)_{\lambda \in L_1}) && [\text{def. } p_{L_2,L_1}] \\ &= (x_\lambda)_{\lambda \in L_0} && [\text{def. } p_{L_1,L_0}] \\ &= p_{L_2,L_0}((x_\lambda)_{\lambda \in L_2}) && [\text{def. } p_{L_2,L_0}]. \end{aligned}$$

Step 3: $\geq$ is reflexive. It suffices to assume that $(L, z) \in \mathcal{X}$ and to prove that $(L, z) \geq (L, z)$. This is true because $L \supseteq L$ and (by step 1) $z = p_{L,L} z$.

Step 4: $\geq$ is transitive. It suffices to assume that $(L_0, z_0), (L_1, z_1), (L_2, z_2) \in \mathcal{X}$, $(L_2, z_2) \geq (L_1, z_1) \geq (L_0, z_0)$, and to prove that $(L_2, z_2) \geq (L_0, z_0)$. For this, we show that $L_2 \supseteq L_0$,

$$\begin{aligned} L_2 &\supseteq L_1 && [(L_2, z_2) \geq (L_1, z_1)] \\ &\supseteq L_0 && [(L_1, z_1) \geq (L_0, z_0)], \end{aligned}$$

and that $z_0 = p_{L_2,L_0} z_2$,

$$\begin{aligned} p_{L_2,L_0} z_2 &= p_{L_1,L_0} p_{L_2,L_1} z_2 && [\text{by step 2}] \\ &= p_{L_1,L_0} z_1 && [(L_2, z_2) \geq (L_1, z_1)] \\ &= z_0 && [(L_1, z_1) \geq (L_0, z_0)]. \end{aligned}$$

Step 5: $\geq$ is antisymmetric. If suffices to assume that $(L_0, z_0), (L_1, z_1) \in \mathcal{X}$, $(L_0, z_0) \geq (L_1, z_1)$ and $(L_1, z_1) \geq (L_0, z_0)$, and to prove that $(L_0, z_0) = (L_1, z_1)$. To show that $L_0 = L_1$, notice that $L_0 \supseteq L_1$ (because $(L_0, z_0) \geq (L_1, z_1)$) and $L_1 \supseteq L_0$ (because $(L_1, z_1) \geq (L_0, z_0)$). To show that $z_0 = z_1$, notice that

$$\begin{aligned} z_0 &= p_{L_1,L_0} z_1 && [(L_1, z_1) \geq (L_0, z_0)] \\ &= p_{L_1,L_1} z_1 && [L_1 = L_0] \\ &= z_1 && [\text{by step 1}]. \end{aligned}$$

□

Exercise 5.3. Let $H \subseteq \mathcal{X}$ be a totally ordered subset. Show that H admits an upper bound in $\mathcal{X}$.

Solution. If H is empty, then the result is true, because any element of $\mathcal{X}$ is an upper bound. So we may assume that H is not empty. Let $L = \bigcup_{(L_0,z_0)\in H} L_0 \subseteq \Lambda$, and define $z \in \prod_{\lambda\in L} X_\lambda$ via $z_\lambda = (z_0)_\lambda$, if $\lambda \in L_0$ with $(L_0, z_0) \in H$. The fact that this is well-defined follows from the condition that H is totally ordered: if λ also satisfies $\lambda \in L_1$, and $(L_1, z_1) \in H$, then assume up to changing roles that $L_0 \subseteq L_1$, $z_0 = p_{L_1,L_0} z_1$. But then $(z_0)_\lambda = (z_1)_\lambda$. This shows that $(L, z) \in \mathcal{X}$, and by definition we have $(L, z) \geq (L_0, z_0)$ for every $(L_0, z_0) \in H$. This means that (L, z) is an upper bound for H in $\mathcal{X}$. □

Exercise 5.4. Let (L, z) be a maximal element in $\mathcal{X}$, which exists by Zorn's Lemma. Show that $L = \Lambda$.

Solution. Assume by contradiction that $L \subsetneq \Lambda$. Take $\lambda \in \Lambda \backslash L$, and $z_\lambda \in X_\lambda$ (here we use that $X_\lambda \neq \varnothing$). Define $L' = L \cup \{\lambda\}$, and $z' \in \prod_{\lambda' \in L'} X_{\lambda'}$ via $z'_{\lambda'} = z_{\lambda'}$ if $\lambda' \neq \lambda$, $z'_\lambda = z_\lambda$. But then $(L', z') \in \mathcal{X}$, $(L', z') > (L, z)$, which is a contradiction since (L, z) is maximal. □

6 Exercise sheet No. 6

Exercise 6.1. Let (X, d) be a metric space. Show that $d' \colon X \times X \to \mathbb{R}$, defined by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \ x, y \in X,$$

is also a metric.

Solution. Non-negativity of d', as well as its symmetry and the fact that it vanishes precisely when $x = y$, all follow immediately from the corresponding property for d. We therefore check the triangle inequality.

Consider the function

$$f \colon [0, +\infty) \to [0, +\infty)$$

defined by

$$f(t) = \frac{t}{1 + t}.$$

Note that $d' = f \circ d$. We have

$$f' = \frac{1}{(1 + t)^2},$$

which is everywhere strictly positive, and so f is strictly increasing.

Let $x, y, z \in X$. Combining this fact with the triangular inequality of d, we have

$$\begin{aligned} d'(x, y) &= f(d(x, y)) \\ &\leq f(d(x, z) + d(z, y)) \\ &= \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq f(d(x, z)) + f(d(z, y)) \\ &= d'(x, z) + d'(z, y). \end{aligned} \tag{2}$$

□

Exercise 6.2. Show that d and d' induce the same topology.

Solution. First, we prove the following general fact:

Fact: A metric topology is characterized by all its convergent sequences.

This means that, if d_1 and d_2 are two metrics on X, then they induce the same topology on X if and only if they have the same convergent sequences (i.e. if $(x_n)_{n \in \mathbb{N}}$ converges in $\mathcal{O}_{d_1}$ to $x \in X$, then it also converges in $\mathcal{O}_{d_2}$ to $x \in X$, and viceversa).

Proof of **Fact**. We prove that the closed sets for $\mathcal{O}_{d_1}$ are the same as the closed sets for $\mathcal{O}_{d_2}$, which by symmetry implies $\mathcal{O}_{d_1} = \mathcal{O}_{d_2}$.

Let C be closed for $\mathcal{O}_{d_1}$, and let x be a limit point of C for $\mathcal{O}_{d_2}$. Since the topology is metric, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset C$ which converges to x in $\mathcal{O}_{d_2}$. By assumption, $(x_n)_{n \in \mathbb{N}} \subset C$ also converges to x in $\mathcal{O}_{d_1}$. But since C is closed for $\mathcal{O}_{d_1}$, we have $x \in C$. Then C contains all its limit points with respect to $\mathcal{O}_{d_2}$, and so it is closed with respect to $\mathcal{O}_{d_2}$. □

Using the above fact, it suffices to show that d and d' have the same convergent sequences. Assume that $(x_n)_{n\in\mathbb{N}}$ converges in $\mathcal{O}_d$ to $x \in X$. Then

$$d'(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} \leq d(x_n, x) \to 0,$$

as $n \to +\infty$, and so x_n converges in $\mathcal{O}_{d'}$ to x. Reciprocally, assume that $(x_n)_{n\in\mathbb{N}}$ converges in $\mathcal{O}_{d'}$ to $x \in X$. This implies that $d(x_n, x)$ is bounded: indeed, if it is not, there exists a subsequence x_{n_k} for which $d(x_{n_k}, x) \to +\infty$ as $k \to +\infty$. But then $d(x_{n_k}, x) \to 1$ as $k \to +\infty$, which is absurd. Then we may find $M > 0$ such that $d(x_n, x) \leq M$ for all n. Moreover,

$$\frac{d(x_n, x)}{1 + M} \leq \frac{d(x_n, x)}{1 + d(x_n, x)} = d'(x_n, x) \to 0,$$

as $n \to +\infty$. This implies that $(x_n)_{n\in\mathbb{N}}$ converges in $\mathcal{O}_d$ to $x \in X$, and finishes the proof. $\square$

Exercise 6.3. For each $i \in \mathbb{N}$, let (X_i, d_i) be a metric space. For each i define a metric d'_i as in exercise 6.1. Let $X = \prod_{i\in\mathbb{N}} X_i$. Define

$$d\colon X \times X \longrightarrow \mathbb{R}$$

for $x = (x_i)_{i\in\mathbb{N}} \in X$ and $y = (y_i)_{i\in\mathbb{N}} \in X$ by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d'_i(x_i, y_i).$$

Show that d is well defined, that is, that the sum above converges and that d defines a metric on X.

Solution. We show that the sum $d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d'_i(x_i, y_i)$ converges. For this, note that

$$\begin{aligned} \frac{1}{2^i} d'_i(x_i, y_i) &= \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} \\ &\leq \frac{1}{2^i}, \end{aligned}$$

so each term of this series is smaller than the corresponding term of a geometric series. Since the geometric series converges, the series defining $d(x, y)$ converges as well.

We show that d is a metric.

d is a real valued non-negative function: d is a convergent sum of positive terms.

d is non-degenerate, i.e. $\forall x, y \in X\colon d(x, y) = 0 \iff x = y$:

$$\begin{aligned} d(x, y) = 0 &\iff \sum_{i=1}^{\infty} d'_i(x_i, y_i) = 0 && [\text{def. } d(x, y)] \\ &\iff \forall i \in \mathbb{N}\colon d'_i(x_i, y_i) = 0 && [\text{sum of} > 0 \text{ terms is 0 iff each term is 0}] \\ &\iff \forall i \in \mathbb{N}\colon x_i = y_i && [d'_i \text{ is a metric on } X_i] \\ &\iff x = y && [\text{def. of } \textstyle\prod_{i=1}^{\infty} X_i]. \end{aligned}$$

d is symmetric, i.e. $\forall x, y \in X\colon d(x,y) = d(y,x)$:

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{\infty} \frac{1}{2^i} d_i'(x_i, y_i) && [\text{def. } d(x,y)] \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} d_i'(y_i, x_i) && [d_i'(y_i,x_i) \text{ is symmetric}] \\ &= d(y,x) && [\text{def. } d(y,x)]. \end{aligned}$$

d satisfies the triangle inequality, i.e. $\forall x, y, z \in X\colon d(x,z) \le d(x,y) + d(y,z)$:

$$\begin{aligned} d(x,z) &= \sum_{i=1}^{\infty} \frac{1}{2^i} d_i'(x_i, z_i) && [\text{def. of } d(x,z)] \\ &\le \sum_{i=1}^{\infty} \frac{1}{2^i} (d_i'(x_i, y_i) + d_i'(y_i, z_i)) && [d_i' \text{ satisfies the triangle inequality}] \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} d_i'(x_i, y_i) + \sum_{i=1}^{\infty} \frac{1}{2^i} d_i'(y_i, z_i) \\ &= d(x,y) + d(y,z) && [\text{def. of } d(x,y)]. \end{aligned}$$

□

Exercise 6.4. Let d be the metric on $X = \prod_{i\in\mathbb{N}} X$ from exercise 6.3. Show that the topology $\mathcal{O}_d$ on X induced from d coincides with the product topology.

Solution. Denote by $\mathcal{B}$ the basis of the product topology on X. Denote by $\mathcal{O}_\mathcal{B}$ the product topology on X. We need to show that $\mathcal{O}_d = \mathcal{O}_\mathcal{B}$.

We show that $\mathcal{O}_d \subset \mathcal{O}_\mathcal{B}$.

Step 1: we show that it suffices to assume that $x \in X$, $r > 0$, $y \in B_r(x)$, and prove that there exists $B_y \in \mathcal{B}$ such that $y \in B_y \subset B_r(x)$. For this, let U be open in $\mathcal{O}_d$. Then, for each $x \in U$ there exists $r_x > 0$ such that $x \in B_{r_x}(x) \subset U$. Then,

$$\begin{aligned} U &= \bigcup_{x\in U} B_{r_x}(x) && [\forall x \in U\colon x \in B_{r_x}(x) \subset U] \\ &= \bigcup_{x\in U} \bigcup_{y \in B_{r_x}(x)} B_y && [\forall y \in B_{r_x}(x)\colon y \in B_y \subset B_{r_x}(x)]. \end{aligned}$$

Each B_y is in $\mathcal{O}_\mathcal{B}$ by definition of $\mathcal{O}_\mathcal{B}$, and since $\mathcal{O}_\mathcal{B}$ is a topology U is also in $\mathcal{O}_\mathcal{B}$.

Step 2: we define B_y. Define $\varepsilon = r - d(x,y) > 0$. There exists an $N \in \mathbb{N}$ such that $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}$, since the sum $\sum_{i=1}^{\infty} \frac{1}{2^i}$ converges. There exists an r' such that $\sum_{i=1}^{N} \frac{1}{2^i} \frac{r'}{1+r'} < \varepsilon$, because the map $(-1, +\infty) \longrightarrow \mathbb{R}$ given by $x \longmapsto \frac{x}{1+x}$ is smooth, strictly increasing, and maps 0 to 0. Then define $U_i^y = B_{r'}(y_i)$ if $i \le N$ and $U_i^y = X_i$ if $i > N$. Define $B_y = \prod_{i\in\mathbb{N}} U_i^y$. Then, by definition of $\mathcal{B}$, $B \in \mathcal{B}$.

Step 3: we show that $y \in B \subset B_r(x)$. It's immediate that $y \in B$. For $B \subset B_r(x)$, it suffices to assume that $z \in B$ and to prove that $z \in B_r(x)$. Notice that since the function $x \longmapsto \frac{x}{1+x}$ is strictly increasing and $\forall i \le N\colon d_i(y_i, z_i) < r'$, we have that $\forall i \le N\colon \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)} < \frac{r'}{1+r'}$.

$$\begin{aligned} d(x,z) &\le d(x,y) + d(y,z) \\ &= d(x,y) + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(y_i,z_i)}{1 + d_i(y_i,z_i)} \end{aligned}$$

$$
\begin{aligned}
&= d(x,y) + \sum_{i=1}^{N} \frac{1}{2^i} \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)} \\
&\leq d(x,y) + \sum_{i=1}^{N} \frac{1}{2^i} \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \\
&< d(x,y) + \sum_{i=1}^{N} \frac{1}{2^i} \frac{r'}{1+r'} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \\
&< d(x,y) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= d(x,y) + \varepsilon \\
&= r.
\end{aligned}
$$

We show that $\mathcal{O}_d \supset \mathcal{O}_{\mathcal{B}}$.

Step 1: we show that it suffices to assume that $B \in \mathcal{B}$, $x \in B$, and to prove that there exists $r_x > 0$ such that $B_{r_x}(x) \subset B$. For this, let U be open in X with respect to the $\mathcal{O}_{\mathcal{B}}$. Then, there exists a collection $\{B_j\}_{j \in I} \subset \mathcal{B}$ such that $U = \bigcup_{j \in J} B_j$.

$$
\begin{aligned}
U &= \bigcup_{j \in J} B_j \\
&= \bigcup_{j \in J} \bigcup_{x \in B_j} B_{r_x}(x) \quad [\forall x \in B \colon x \in B_{r_x}(x) \subset B].
\end{aligned}
$$

Each $B_{r_x}(x)$ is in $\mathcal{O}_d$ by definition of $\mathcal{O}_d$, and since $\mathcal{O}_d$ is a topology U is also in $\mathcal{O}_d$.

Step 2: we define r_x. By definition of basis for the product topology, there exists $\{U_i\}_{i \in \mathbb{N}}$ such that $U_i \subset X_i$ is open, $U_i \neq X_i$ only for finitely many i's, and $B = \prod_{i \in \mathbb{N}} U_i$. There exists an $N \in \mathbb{N}$ such that for all $i > N$ we have $U_i = X_i$. For each $i \leq N$, since each U_i is open and X_i is a metric space, there exists $r_i > 0$ such that $B_{r_i}(x_i) \subset U_i$. Define $r' = \min\{r_1, \ldots, r_N\}$ and $r_x = \frac{1}{2^N} \frac{r'}{1+r'}$.

Step 3: we show that $B_{r_x}(x) \subset B$. For this, it suffices to assume that $y \in B_{r_x}(x)$ and to prove $y \in B$. For this, it suffices to show $\forall i = 1, \ldots, N \colon y_i \in B_{r'}(x_i)$, because then $\forall i = 1, \ldots, N \colon y_i \in B_{r'}(x_i) \subset B_{r_i}(x_i) \subset U_i$.

$$
\begin{aligned}
y \in B_{r_x}(x) &\iff d(x,y) < r_x \\
&\iff \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} < r_x \\
&\implies \forall i = 1, \ldots, N \colon \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} < 2^i r_x \leq 2^N r_x = \frac{r'}{1+r'} \\
&\implies \forall i = 1, \ldots, N \colon d_i(x_i,y_i) < r',
\end{aligned}
$$

where the last implication follows because $(-1, +\infty) \longrightarrow (-\infty, 1), x \longmapsto \frac{x}{1+x}$ is a strictly increasing smooth bijection. $\square$

7 Exercise sheet No. 7

Exercise 7.1. Show that homotopy is an equivalence relation on the set of continuous functions from X to Y.

Solution. For $f, g\colon X \longrightarrow Y$ continuous, write $f \simeq g$ when f is homotopic to g, i.e. there exists $H\colon [0,1]\times X \longrightarrow Y$ continuous such that $\forall x \in X\colon H(0,x) = f(x), H(1,x) = g(x)$.

Reflexive: it suffices to assume that $f\colon X \longrightarrow Y$ is continuous, and prove that f is homotopic to f. Define $H\colon [0,1] \times X \longrightarrow Y$ by $H(t,x) = f(x)$. Then, H is continuous and $\forall x \in X\colon H(0,x) = f(x), H(1,x) = f(x)$. By definition of homotopy, f is homotopic to f.

Symmetric: it suffices to assume that $f, g\colon X \longrightarrow Y$ are continuous, that $f \simeq g$, and to prove that $g \simeq f$. Since $f \simeq g$, there exists $H'\colon [0,1] \times X \longrightarrow Y$ continuous such that $\forall x \in X\colon H'(0,x) = f(x), H'(1,x) = g(x)$. Define $H\colon [0,1]\times X \longrightarrow Y$ by $H(t,x) = H'(1-t,x)$. Then H is continuous and $\forall x \in X\colon H(0,x) = g(x), H(1,x) = f(x)$. By definition of homotopic, $g \simeq f$.

Transitive: it suffices to assume that $f, g, h\colon X \longrightarrow Y$ are continuous, that $f \simeq g$ and $g \simeq h$, and to prove that $f \simeq h$. Since $f \simeq g$, there exists $H_1\colon [0,1] \times X \longrightarrow Y$ continuous such that $\forall x \in X\colon H_1(0,x) = f(x), H_1(1,x) = g(x)$. Since $g \simeq h$, there exists $H_2\colon [0,1] \times X \longrightarrow Y$ continuous such that $\forall x \in X\colon H_2(0,x) = g(x), H_2(1,x) = h(x)$. Define $H\colon [0,1] \times X \longrightarrow Y$ by

$$H(t,x) = \begin{cases} H_1(2t,x) & \text{if } t \in [0,1/2) \\ H_2(2t-1,x) & \text{if } t \in [1/2,1]. \end{cases}$$

Then, H is continuous, since for $t = 1/2$ we have $H_1(21/2,x) = H_1(1,x) = H_2(0,x) = H_2(1-21/2,x)$. Also, $\forall x \in X\colon H(0,x) = f(x), H(1,x) = h(x)$. By definition of homotopy, $f \simeq h$. $\square$

Exercise 7.2. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ and $(Z, \mathcal{O}_Z)$ be three topological spaces and let $f_0, f_1\colon X \longrightarrow Y$ and $g_0, g_1\colon Y \longrightarrow Z$ be two pairs of homotopic continuous functions. Show that $g_0 f_0$ and $g_1 f_1$ are homotopic continuous functions from X to Z.

Solution. Since $f_0 \simeq f_1$, there exists $F\colon [0,1] \times X \longrightarrow Y$ continuous such that $\forall x \in X\colon F(0,x) = f_0(x), F(1,x) = f_1(x)$. Since $g_0 \simeq g_1$, there exists $G\colon [0,1] \times X \longrightarrow Y$ continuous such that $\forall x \in X\colon G(0,x) = g_0(x), G(1,x) = g_1(x)$.

We show that $g_0 f_0 \simeq g_0 f_1$. For this, define $H\colon [0,1] \times X \longrightarrow Z$ by $H(t,x) = g_0(F(t,x))$. Then, H is continuous and $\forall x \in X\colon H(0,x) = g_0 f_0(x), H(1,x) = g_0 f_1(x)$. By definition of homotopic, $g_0 f_0 \simeq g_0 f_1$.

We show that $g_0 f_1 \simeq g_1 f_1$. For this, define $I\colon [0,1]\times X \longrightarrow Z$ by $I(t,x) = G(t, f_1(x))$. Then, I is continuous and $\forall x \in X\colon I(0,x) = g_0 f_1(x), I(1,x) = g_1 f_1(x)$. By definition of homotopic, $g_0 f_1 \simeq g_1 f_1$.

So, $g_0 f_0 \simeq g_0 f_1 \simeq g_1 f_1$. Since $\simeq$ is an equivalence relation, $g_0 f_0 \simeq g_1 f_1$. $\square$

Exercise 7.3. Show that the notion of homotopy equivalence defines an equivalence relation on the set of topological spaces.

Solution. We write $X \simeq Y$ to denote that X is homotopy equivalent to Y.

Reflexive: $X \simeq X$ is homotopy equivalent to itself, since if $f = g = id\colon X \to X$ is the identity map, then $f \circ g = g \circ f = id$, which is homotopic to id via the constant homotopy $H : [0,1] \times X \to X$, $H(\cdot, x) = x$.

Symmetric: $X \simeq Y$ means that there exist $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$. This definition is clearly symmetric in X and Y.

Transitive: If $X \simeq Y$, $Y \simeq Z$, let $f\colon X \to Y$, $g\colon Y \to X$, $k\colon Y \to Z$, $l\colon Z \to Y$, such that $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$, $k \circ l \simeq id_Z$, $l \circ k \simeq id_Y$. Then

$$(k \circ f) \circ (g \circ l) = k \circ (f \circ g) \circ l \simeq k \circ id_Y \circ l = k \circ l \simeq id_Z.$$

Similarly one checks that $(g \circ l) \circ (k \circ f) \simeq id_X$.

Note that we are using the general fact that if $a \simeq b$, then $a \circ c \simeq b \circ c$ for arbitrary maps a, b, c, and similarly for composition on the left. This is proved as follows: if H is a homotopy between a and b, then $H \circ c$ is a homotopy between $a \circ c$ and $b \circ c$. $\square$

Exercise 7.4. Compute the zero-th Betti number $b_0(X)$ in the case where X is an arbitrary set with the discrete topology, or with the trivial topology.

Solution. We consider the case of the discrete topology, where $\mathcal{O}_X = P(X)$ is the powerset of X. In this case, we prove:

Claim: A map $\gamma : Y \to X$ from a compact and connected space Y is continuous if and only if it is constant.

Indeed, we have

$$Y = \bigcup_{x \in \gamma(Y)} \gamma^{-1}(\{x\}).$$

Since $\{x\}$ is open for all $x \in X$ and γ is continuous, we have $\gamma^{-1}(\{x\})$ is open for all $x \in \gamma(Y)$. Using compactness of Y, we find finitely many $x_1, \ldots, x_n \in \gamma(Y) \subset X$ for which

$$Y = \bigcup_{i=1}^{n} \gamma^{-1}(\{x_i\}).$$

Consider

$$I = \{i > 1 : \gamma^{-1}(\{x_1\}) \cap \gamma^{-1}(\{x_i\}) \neq \varnothing\}.$$

Note that $x_i = x_1$ for every $i \in I$, by definition of I. We now reorder all indices $i > 1$, so that $I = \{1, \ldots, k\}$, and $i \notin I$ for $i > k$. If $k = n$, we are done, since $x_i = x_1$ for all $i \in I$ and so $\bigcup_{i=1}^{k} \gamma^{-1}(\{x_i\}) = \gamma^{-1}(\{x_1\})$. But if $k < n$, then

$$Y = \gamma^{-1}(\{x_1\}) \bigcup_{i>k} \gamma^{-1}(\{x_i\})$$

is the disjoint union of two open sets, which contradicts the assumption that Y is connected. This proves the claim.

Setting $Y = [0,1]$ in the above claim, which is compact and connected, we obtain that the path connected components of X are of the form $\{x\}$ for $x \in X$, i.e. $X = X/\sim$. This implies that $b_0(X) = \#X$ is the cardinality of X.

In the case of the trivial topology $\mathcal{O}_X = \{X, \varnothing\}$, we prove:

Claim: *Every* map $\gamma : Y \to X$ is continuous for any topological space Y.

Indeed, an open set $U \subset X$ is either $U = X$ (and so $\gamma^{-1}(U) = Y$ is open) or $U = \varnothing$ (and so $\gamma^{-1}(U) = \varnothing$ is also open). This proves the second claim.

Setting $Y = [0, 1]$, this implies that there is only one path component in X, since for $x, y \in X$ we can just take *any* map $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, and so $x \sim y$. This means that $X/\sim$ contains a single point, and so $b_0(X) = 1$. $\square$

8 Exercise sheet No. 8

Exercise 8.1. Compute the zeroth Betti number of S^n for all $n \in \mathbb{N}_0$.

Solution. We show that $b_0(S^0) = 2$. Notice that $S^0 = \{x \in \mathbb{R} | \|x\| = 1\} = \{-1, 1\}$. Denote by C_1 the path component of S^0 containing 1. Then, $C = \{1\}$ or $C = \{-1, 1\}$. C equals $\{-1, 1\}$ if and only if there exists a continuous map $\gamma\colon [0, 1] \longrightarrow \{-1, 1\}$ such that $\gamma(0) = 1$ and $\gamma(1) = -1$, which is false. So $C_1 = \{1\}$. By a similar argument, the path component of S^0 containing -1 is C_{-1}. Then,

$$\begin{aligned} b_0(S^0) &= \#(S^0/\sim) && \text{[def. of } b_0] \\ &= \#\{\{-1\}, \{1\}\} \\ &= 2. \end{aligned}$$

We show that $\forall n \geq 1\colon b_0(S^n) = 1$. This is equivalent to S^n being path connected. So, it suffices to assume that $x \in S^n$ and to prove that there exists a continuous map $\gamma\colon [0, 1] \longrightarrow S^n$ continuous such that $\gamma(0) = N \coloneqq (1, 0, \ldots, 0)$ and $\gamma(1) = x$. In the case where $x = N$, let γ be the constant map at $x = N$. In the case where $x = S = (-1, 0, \ldots, 0) \in S^n \subset \mathbb{R}^{n+1}$, define $\gamma(t) = (\cos(\pi t), \sin(\pi t), 0, \ldots, 0) \in S^n \subset \mathbb{R}^{n+1}$. It remains to prove the result in the case where $x \notin \{N, S\}$. In this case, there exists a rotation matrix $A \in \mathrm{SO}(n+1)$ such that $A(N) = N$ and such that $A^{-1}(x)$ is of the form $A^{-1}(x) = (\cos(\pi s), \sin(\pi s), 0, \ldots, 0)$, for some $s \in [0, 2)$. Then, define

$$\begin{aligned} \gamma'\colon [0, 1] &\longrightarrow S^n \\ t &\longmapsto (\cos(\pi t s), \sin(\pi t s), 0, \ldots, 0), \end{aligned}$$

and $\gamma = A \circ \gamma'$. Then, γ is continuous, $\gamma(0) = N$ and $\gamma(1) = x$. $\square$

Exercise 8.2. Show that for $n > 1$ $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$.

Solution. We start by showing that for every $n \in \mathbb{N}_0$, S^n is homotopy equivalent to $\mathbb{R}^{n+1} \setminus \{0\}$. For this, define

$$\begin{aligned} g\colon \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow S^n & \iota\colon S^n &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ x &\longmapsto \frac{x}{\|x\|}, & x &\longmapsto x. \end{aligned}$$

Then, g and ι are continuous and $g\iota = \mathrm{id}_{S^n}$. We show that ιg is homotopic to $\mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$. For this, define

$$\begin{aligned} H\colon [0, 1] \times \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ (t, x) &\longmapsto t\iota g(x) + (1 - t)x. \end{aligned}$$

The following computations show that H is well defined:

$$\begin{aligned} t\iota g(x) + (1 - t)x &= t\frac{x}{\|x\|} + (1 - t)x && \text{[def. of } \iota, g] \\ &= \Big(t\Big(\frac{1}{\|x\|} - 1\Big) + 1\Big)x, \end{aligned}$$

$$
\begin{aligned}
\|x\| > 0 &\Longrightarrow \frac{1}{\|x\|} > 0 \\
&\Longrightarrow \frac{1}{\|x\|} - 1 > -1 \\
&\Longrightarrow t\Big(\frac{1}{\|x\|} - 1\Big) > -t \\
&\Longrightarrow t\Big(\frac{1}{\|x\|} - 1\Big) > -1 \qquad\qquad [t \in [0,1]] \\
&\Longrightarrow \Big(t\Big(\frac{1}{\|x\|} - 1\Big) + 1\Big) > 0.
\end{aligned}
$$

Also, H is continuous and for every x, $H(0,x) = x$, $H(1,x) = \iota g(x)$. So, the maps ιg and $\mathrm{id}_{\mathbb{R}^{n+1}\setminus\{0\}}$ are homotopic, and S^n is homotopy equivalent to $\mathbb{R}^{n+1} \setminus \{0\}$.

To show that for $n > 1$ $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$, we assume by contradiction that $n > 1$ and that there exists $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ a homeomorphism. Then, $\phi|_{\mathbb{R}^n\setminus\{0\}}\colon \mathbb{R}^n\setminus\{0\} \longrightarrow \mathbb{R} \setminus \{\phi(0)\}$ is a homeomorphism. The following gives the desired contradiction (in the next computation we denote homotopy equivalence by $\simeq$ and homeomorphism by $\cong$):

$$
\begin{aligned}
2 &= b_0(S^0) && [\text{by exercise 8.1}] \\
&= b_0(\mathbb{R} \setminus \{0\}) && [S^0 \simeq \mathbb{R} \setminus \{0\} \ \text{ and theorem 2.2 in lecture notes 7}] \\
&= b_0(\mathbb{R} \setminus \{\phi(0)\}) && [\mathbb{R} \setminus \{0\} \cong \mathbb{R} \setminus \{\phi(0)\} \text{ and theorem 2.2. in lecture notes 7}] \\
&= b_0(\mathbb{R}^n \setminus \{0\}) && [\mathbb{R}^n \setminus \{0\} \cong \mathbb{R} \setminus \{\phi(0)\}, \text{ by our assumption}] \\
&= b_0(S^{n-1}) && [S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \text{ and theorem 2.2 in lecture notes 7}] \\
&= 1 && [\text{by exercise 8.1}].
\end{aligned}
$$

□

Exercise 8.3. Draw an i-simplex for $i = 0, 1, 2$. Show that the i-simplex is homeomorphic to a point ($i = 0$), an interval ($i = 1$), a triangle ($i = 2$) and a tetrahedron ($i = 3$).

Solution. Recall that $\Delta^i = \left\{ \sum_{j=1}^{i+1} t_j e_j : 0 \le t_j \le 1, \sum_{j=1}^{i+1} t_j = 1 \right\}$. Note that for $i = 0$, this is just the singleton $\Delta^0 = \{e_1\}$.

To understand the cases $i > 0$, consider the projection map

$$p : \mathbb{R}^{i+1} \to \mathbb{R}^i$$

$$(t_1, \ldots, t_{i+1}) \mapsto (t_1, \ldots, t_i),$$

where we identify the point $\sum_{j=1}^{i+1} t_j e_j$ with the tuple $(t_1, \ldots, t_{i+1})$. The image of $\Delta^i \subset \mathbb{R}^{i+1}$ is precisely

$$p(\Delta^i) = \left\{ \sum_{j=1}^{i} t_j e_j : 0 \le t_j \le 1, \sum_{j=1}^{i} t_j \le 1 \right\}.$$

For $i = 1, 2, 3$, this is respectively an interval, a triangle, and a tetrahedron (see Figure 1). Moreover, the restriction $p|_{\Delta^i} : \Delta^i \to p(\Delta^i)$ is a (linear) homeomorphism, with inverse map

$$p^{-1} : p(\Delta^i) \to \Delta^i$$

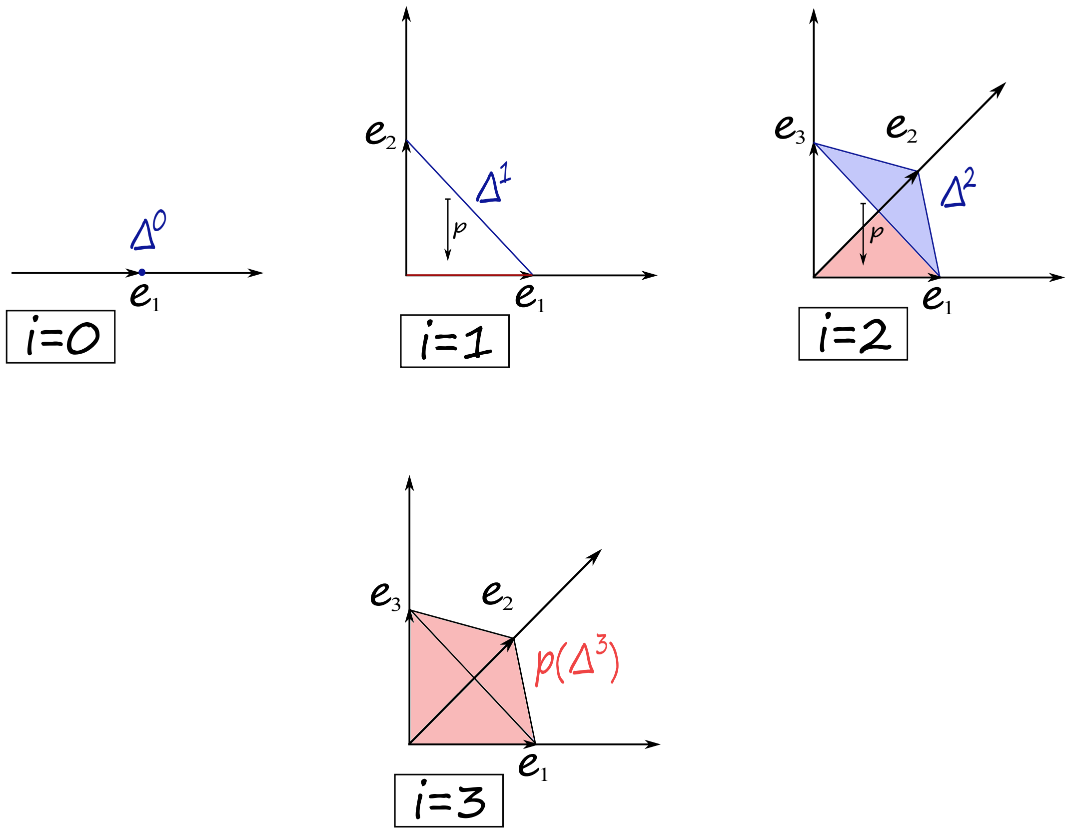

Figure 1: The i-simplex Δ^i is depicted in blue, and its image under the projection map p is depicted in red.

$$(t_1, \ldots, t_i) \mapsto (t_1, \ldots, t_i, 1 - \sum_{j=1}^{i} t_j).$$

$\square$

Exercise 8.4. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homeomorphic topological spaces. Show that $b_n(X) = b_n(Y)$ for every n.

Solution. In the language of category theory, we first show that taking the homology of a topological space is *functorial*. This means that not only we can associate the vector space $H_n(X)$ for every topological space X, but we may also associate a linear map $f_* : H_n(X) \to H_n(Y)$ for every continuous map $f : X \to Y$, which preserves compositions in the sense that $(f \circ g)_* = f_* \circ g_*$ (this means that H_n is *covariant* as a functor), and maps the identity to the identity, i.e. $(id)_* = id$.

Indeed, given a continuous map $f : X \to Y$, and given any n, we start at the *chain-level*, by defining the map

$$f_* : C_n(X) \to C_n(Y)$$

first by prescribing that $f_*(\phi) := f \circ \phi$ for every basis element ϕ (a continuous map $\phi : \Delta^n \to X$), and then extending to $C_n(X)$ as the unique possible way of getting a *linear* extension, i.e. via

$$f_*\left(\sum_j r_j\phi_j\right) = \sum_j r_j f_*(\phi_j) = \sum_j r_j(f \circ \phi_j).$$

In order to have a well-defined map in homology, we need to check that f_* commutes with the differential, i.e. that

$$d_n \circ f_* = f_* \circ d_n.$$

Note that both sides are linear, so it suffices to check the equality on basis elements ϕ. For this we observe that $\partial_n^j(f \circ \phi) = f \circ \partial_n^j\phi$, for every basis element ϕ, simply because $\partial_n^j\phi$ is the restriction of ϕ to the j-th face of Δ^n. Then

$$(d_n \circ f_*)\phi = -\sum_{j=1}^{n+1}(-1)^j\partial_n^j(f \circ \phi) = -\sum_{j=1}^{n+1}(-1)^j f \circ \partial_n^j\phi = (f_* \circ d_n)\phi.$$

From the above equation, we see that f_* maps cycles to cycles (i.e. it preserves $\ker d_n$) and it maps boundaries to boundaries (i.e. it preserves im d_n), and so we obtain a linear map

$$f_* : H_n(X) \to H_n(Y),$$

at the level of homology, which is defined as

$$f_*[\phi] = [f_*(\phi)] = [f \circ \phi],$$

where $[\phi]$ denotes the homology class of a basis element ϕ (and f_* is uniquely extended to a linear map).

It is now easy to check that we have the equations

$$(f \circ g) = f_* \circ g_*,\ (id)_* = id,$$

for any pairs of maps f, g, which follows from the same equation at the chain level.

If f is a homeomorphism, the functorial properties of H_n immediately imply that f_* is an isomorphism. Indeed, if g is its inverse, then

$$f_* \circ g_* = (f \circ g)_* = (id)_* = id,$$

and similarly $g_* \circ f_* = id$. This means that g_* is the (linear) inverse of f_*.

In particular, if X and Y are homeomorphic, then their homologies $H_n(X)$ and $H_n(Y)$ are isomorphic as vector spaces (for every n), and so their ranks (the Betti numbers) are the same:

$$b_n(X) = \dim H_n(X) = \dim H_n(Y) = b_n(Y).$$

□

9 Exercise sheet No. 9

Exercise 9.1. For $n \in \mathbb{N}_0$, show that

$$H_n(X) = \bigoplus_{[x]\in X/\sim} H_n([x]).$$

Solution. For a set S, denote by $F(S)$ the free vector space over $\mathbb{R}$ generated by the elements of S. Denote by $C^0(Y, Z)$ the set of continuous maps from a topological space Y to a topological space Z. So, using this notation, $C_n(X) = F(C^0(\Delta^n, X))$.

First, note that

$$\begin{aligned}
C_n(X) &= F(C^0(\Delta^n, X)) && [\text{def. of } C_n(X)] \\
&= F\Big(C^0\Big(\Delta^n, \bigcup_{[x]\in X/\sim}[x]\Big)\Big) && [\sim \text{ is an equivalence relation}] \\
&= F\Big(\bigcup_{[x]\in X/\sim} C^0\Big(\Delta^n, [x]\Big)\Big) && [\star] \\
&= \bigoplus_{[x]\in X/\sim} F\Big(C^0(\Delta^n, [x])\Big) && [\dagger] \\
&= \bigoplus_{[x]\in X/\sim} C_n([x]) && [\text{def. of } C_n([x])].
\end{aligned}$$

Here, $\star$ is because $C^0\Big(\Delta^n, \bigcup_{[x]\in X/\sim}[x]\Big) = \bigcup_{[x]\in X/\sim} C^0\Big(\Delta^n, [x]\Big)$. We prove each inclusion of this equality of sets. $(\subset)$: If $\phi \in C^0(\Delta^n, \bigcup_{[x]\in X/\sim}[x])$, then since ϕ is continuous we have that $\exists![x] \in X/\sim\ \phi(\Delta^n) \subset [x]$. So, $s \in C^0(\Delta^n, [x]) \subset \bigcup_{[x]\in X/\sim} C^0(\Delta^n, [x])$. $(\supset)$: If $\phi \in \bigcup_{[x]\in X/\sim} C^0(\Delta^n, [x])$, then by definition of disjoint union there exists an $[x] \in X/\sim$ such that $\phi \in C^0(\Delta^n, [x])$. Then, $\phi\colon \Delta^n \longrightarrow [x] \hookrightarrow \bigcup_{[x]\in X/\sim}[x]$, so $\phi \in C^0(\Delta^n, \bigcup_{[x]\in X/\sim}[x])$.

And $\dagger$ is seen to be true by the following computation. For every family of sets $\{S_i\}_{i\in I}$,

$$\begin{aligned}
F(\bigcup_{i\in I} S_i) &= \bigoplus_{a\in\bigcup_{i\in I} S_i} F(\{a\}) \\
&= \bigoplus_{i\in I}\bigoplus_{a\in S_i} F(\{a\}) \\
&= \bigoplus_{i\in I} F(S_i).
\end{aligned}$$

For each $n \in \mathbb{N}_0$, d_n is a linear map $C_n(X) \longrightarrow C_{n-1}(X)$. By the previous equality, d_n is a linear map

$$d_n\colon \bigoplus_{[x]\in X/\sim} C_n([x]) \longrightarrow \bigoplus_{[x]\in X/\sim} C_{n-1}([x]).$$

Denote by $d_{n,[x]}$ the differential of the chain complex $C([x])$:

$$d_{n,[x]}\colon C_n([x]) \longrightarrow C_{n-1}([x]).$$

Then, by definition of d_n, d_n respects this decomposition, in the sense that for each $[x] \in X/\sim$, d_n maps $C_n([x])$ to $C_{n-1}([x])$. By definition of d_n and $d_{n,[x]}$, d_n restricted to $C_n([x])$ is equal to $d_{n,[x]}$. Then, it's possible to show $\ker d_n = \bigoplus_{[x]\in X/\sim} \ker d_{n,[x]}$, $\operatorname{im} d_n = \bigoplus_{[x]\in X/\sim} \operatorname{im} d_{n,[x]}$ and $\operatorname{im} d_{n+1,[x]} \subset \ker d_{n,[x]}$.

Now the result follows from the following computation:

$$\begin{aligned}
H_n(X) &= \frac{\ker d_n}{\operatorname{im} d_{n+1}} && \text{[by def. of } H_n\text{]}\\
&= \frac{\bigoplus_{[x]\in X/\sim} \ker d_{n,[x]}}{\bigoplus_{[x]\in X/\sim} \operatorname{im} d_{n+1,[x]}} && \text{[previous paragraph]}\\
&= \bigoplus_{[x]\in X/\sim} \frac{\ker d_{n,[x]}}{\operatorname{im} d_{n+1,[x]}} && [\clubsuit]\\
&= \bigoplus_{[x]\in X/\sim} H_n([x]) && \text{[by def. of } H_n\text{]},
\end{aligned}$$

where $\clubsuit$ follows from the following general fact about linear algebra. If I is a set and if for every $i \in I$, V_i is a vector space over $\mathbb{R}$ and W_i is a subspace of V_i, then

$$\frac{\bigoplus_{i\in I} V_i}{\bigoplus_{i\in I} W_i} = \bigoplus_{i\in I} \frac{V_i}{W_i}.$$

To prove this fact, consider the following commutative diagram:

$$\begin{tikzcd}
W_i \arrow[r, "\iota_i"] \arrow[d, "\iota_i^W"'] & V_i \arrow[r, "\pi_i"] \arrow[d, "\iota_i^V"] & \frac{V_i}{W_i} \arrow[d, "\iota_i^{V/W}"] \\
\bigoplus_{i\in I} W_i \arrow[r, "\iota"'] & \bigoplus_{i\in I} V_i \arrow[r, "\exists!\phi"'] \arrow[d, "\pi"'] & \bigoplus_{i\in I} \frac{V_i}{W_i} \\
& \frac{\bigoplus_{i\in I} V_i}{\bigoplus_{i\in I} W_i} \arrow[ur, "\exists!\overline{\phi}"'] &
\end{tikzcd} .$$

In this diagram, ι_i is the inclusion, π_i is the projection coming from the quotient of V_i with W_i, ι_i^W, ι_i^V and $\iota_i^{V/W}$ are the inclusions in the direct sums, and π is the projection coming from the quotient of $\bigoplus_{i\in I} V_i$ with $\bigoplus_{i\in I} W_i$. By the universal property of the direct sum, there exists a unique ϕ making the right square commute. It remains to show that $\ker \phi = \bigoplus_{i\in I} W_i$, because in this case, by the universal property of the quotient, there exists a unique linear map $\overline{\phi}\colon \frac{\bigoplus_{i\in I} V_i}{\bigoplus_{i\in I} W_i} \longrightarrow \bigoplus_{i\in I} \frac{V_i}{W_i}$ which is an isomorphism. For this, notice that for every $(v_i)_{i\in I} \in \bigoplus_{i\in I} V_i$

$$\begin{aligned}
(v_i)_{i\in I} \in \ker\phi &\iff \phi((v_i)_{i\in I}) = 0 && \text{[def. ker]}\\
&\iff (\phi\iota_i^V(v_i))_{i\in I} = 0 && \text{[def. } \phi \text{ coming from univ. prop.]}\\
&\iff (\iota_i^{V/W}\pi_i(v_i))_{i\in I} = 0 && \text{[the diagram commutes]}\\
&\iff \forall i \in I\colon \iota_i^{V/W}\pi_i(v_i) = 0 && \\
&\iff \forall i \in I\colon \pi_i(v_i) = 0 && [\ker \iota_i^{V/W} = 0]
\end{aligned}$$

$$\iff \forall i \in I\colon v_i \in W_i \qquad [\ker \pi_i = W_i]$$
$$\iff (v_i)_{i\in I} \in \bigoplus_{i\in I} W_i. \qquad \square$$

Exercise 9.2. Define singular homology using cubes, rather than simplices. Compute the (cubical) singular homology of a point.

Solution. Given a topological space X and $n \in \mathbb{N}_0$, we let $C_n^\square(X)$ be the free $\mathbb{R}$-vector space generated by continuous maps $\phi : I^n \to X$, where $I^n := [0,1]^n$ is the n-dimensional cube. We call such a map ϕ a *n-cube* in X, and elements in $C_n^\square(X)$, *cubical n-chains* in X. We define the following face operators:

$$I_{i,0}^n, I_{i,1}^n\colon I^{n-1} \to I^n,\ 1 \le i \le n,$$

$$I_{i,0}^n(a_1,\dots,a_{n-1}) = (a_1,\dots,a_{i-1},0,a_i,\dots,a_{n-1})$$
$$I_{i,1}^n(a_1,\dots,a_{n-1}) = (a_1,\dots,a_{i-1},1,a_i,\dots,a_{n-1}).$$

Given an n-cube $\phi\colon I^n \to X$, $1 \le i \le n$ and $k \in \{0,1\}$, we denote

$$\phi_{i,k} = \phi \circ I_{i,k}^n\colon I^{n-1} \to X$$

its restriction to the (i,k)-th face. This operation extends linearly to $C_n^\square(X)$.

We now define the operator

$$\partial\colon C_n^\square(X) \to C_{n-1}^\square(X)$$

as the unique linear map which satisfies

$$\partial\phi = \sum_{i=1}^n \sum_{k=0,1} (-1)^{i+k}\phi_{i,k}$$

for every n-cube ϕ in X.

We now check that ∂ defines a differential. First, one easily checks that if $1 \le i \le j \le n-1$ and $k, l \in \{0,1\}$, then

$$I_{i,k}^n \circ I_{j,l}^{n-1} = I_{j+1,l}^n \circ I_{i,k}^{n-1}. \tag{3}$$

This implies that

$$(\phi_{i,k})_{j,l} = (\phi_{j+1,l})_{i,k} \tag{4}$$

for any $(n-1)$-cube ϕ. Moreover, we have

$$\begin{aligned}\partial\partial\phi &= \partial\left(\sum_{i=1}^n \sum_{k=0,1} (-1)^{i+k}\phi_{i,k}\right)\\ &= \sum_{i=1}^n \sum_{k=0,1} \sum_{j=1}^{n-1} \sum_{l=0,1} (-1)^{i+k+j+l}(\phi_{i,k})_{j,l}\end{aligned}$$

In this sum, $(\phi_{j+1,l})_{i,k}$ appears with sign $(-1)^{i+k+j+l+1}$, which is the opposite sign for the term $(\phi_{i,k})_{j,l}$. By Equation (4), we see that everything cancels out and we obtain $\partial^2 = 0$. We may therefore define

$$H_n^\square(X) := \ker(\partial\colon C_n^\square(X) \to C_{n-1}^\square(X))/\mathrm{im}(\partial\colon C_{n+1}^\square(X) \to C_n^\square(X))$$

to be the homology of ∂ in degree n.

We now compute the cubical homology of a point $X = \{pt\}$. Note that there is unique n-cube $\phi_n : I^n \to X$, and so $\phi_{i,k} = \phi_{n-1}$ for every i, k. Therefore

$$\partial\phi_n = \sum_{i=1}^{n} \left(\sum_{k=0,1} (-1)^{i+k} \phi_{n-1} \right) = 0$$

since the term for $\phi_{i,0}$ cancels out that for $\phi_{i,1}$. We obtain

$$H_n^{\square}(X) = \mathbb{R}$$

for *every* $n \in \mathbb{N}_0$, generated by ϕ_n.

Remark. Note that this differs from the computation using simplices, since this homology is non-zero in every degree (this is basically the fact that the standard simplex Δ^n has an *odd* number of faces for $n \geq 2$). This is remedied by modding out degenerate chains, those n-chains which are the restriction of a $(n+1)$-chain to some face, i.e. which lie in the image of a face operator. More on this in the coming weeks. $\square$

10 Exercise sheet No. 10

Exercise 10.1. Let $(X, \mathcal{O})$ be a topological space and $\tilde{Q}_n(X)$ be the real vector space spanned by the continuous maps $q\colon [0,1]^n \longrightarrow X$, and $D_n(X)$ be the subspace spanned by the degenerate maps. Here q is degenerate if there is an i such that $q(x_1, \ldots, x_n)$ does not depend on x_i. Now define the quotient space $Q_n(X) \coloneqq \tilde{Q}_n(X)/D_n(X)$. Show that the boundary operator $\tilde{d}_n\colon \tilde{Q}_n(X) \longrightarrow \tilde{Q}_{n-1}(X)$ that you have defined in the second exercise of last week defines a boundary operator $d_n\colon Q_n(X) \longrightarrow Q_{n-1}(X)$.

Solution. Step 1: we show that $\tilde{d}_n(D_n(X)) \subset D_{n-1}(X)$. To prove this, it suffices to assume that $q\colon I^n \longrightarrow X$ is a continuous and degenerate map (i.e. there exists an $i = 1, \ldots, n$ such that $q(x_1, \ldots, x_n)$ does not depend on x_i) and to prove that $\tilde{d}_n q \in D_{n-1}$. Since

$$\begin{aligned}
\tilde{d}_n q &= \sum_{j=1}^{n} (-1)^j (q \circ I^n_{j,0} - q \circ I^n_{j,1}) && [\text{by def. of } \tilde{d}_n \text{ in exc. 2 of sheet No. 9}] \\
&= \sum_{j \in \{1,\ldots,n\} \setminus \{i\}} (-1)^j (q \circ I^n_{j,0} - q \circ I^n_{j,1}) && [q \text{ doesn't depend on } x_i \Rightarrow q \circ I^n_{j,0} = q \circ I^n_{j,1}],
\end{aligned}$$

it suffices to show that for each $j \in \{1, \ldots, n\} \setminus \{i\}$ both $q \circ I^n_{j,0}$ and $q \circ I^n_{j,1}$ are degenerate cubes. We show that $q \circ I^n_{j,0}$ is a degenerate cube. By definition of degenerate cube, we need to show that there exists a $k = 1, \ldots, n-1$ such that $q \circ I^n_{j,0}(x_1, \ldots, x_{n-1})$ does not depend on x_k. Define $k = i$ if $i < j$ and $k = i - 1$ if $i > j$. Then,

$$q \circ I^n_{j,0}(x_1, \ldots, x_{n-1}) = q(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{n-1}) \qquad [\text{by def. of } I^n_{j,0} \text{ in No. 9}]$$

does not depend on x_k because q does not depend on x_i. The proof of $q \circ I^n_{j,1}$ is degenerate is analogous.

Step 2: we show that there exists a unique $d_n\colon \tilde{Q}_n(X)/D_n(X) \to \tilde{Q}_{n-1}(X)/D_{n-1}(X)$ such that the following diagram commutes:

$$\begin{tikzcd}
D_n(X) \arrow[r, hook, "\iota_n"] \arrow[d, "\tilde{d}_n|_{D_n(X)}"'] & \tilde{Q}_n(X) \arrow[r, "\pi_n"] \arrow[d, "\tilde{d}_n"] & \tilde{Q}_n(X)/D_n(X) \arrow[d, dashed, "\exists! d_n"] \\
D_{n-1}(X) \arrow[r, hook, "\iota_{n-1}"] & \tilde{Q}_{n-1}(X) \arrow[r, "\pi_{n-1}"] & \tilde{Q}_{n-1}(X)/D_{n-1}(X)
\end{tikzcd}.$$

To prove this, by the universal property of the quotient, it suffices to assume that $x \in \ker \pi_n < \tilde{Q}_n(X)$ and to prove that $\pi_{n-1}\tilde{d}_n(x) = 0$.

$$\begin{aligned}
\pi_{n-1}\tilde{d}_n(x) &= \pi_{n-1}\tilde{d}_n\iota_n(x) && [x \in \ker \pi_n = D_n(X)] \\
&= \pi_{n-1}\iota_{n-1}\tilde{d}_n|_{D_n(X)} && [\text{the left square of the diagram commutes}] \\
&= 0 && [\pi_{n-1} \text{ is quotient map} \Longrightarrow \pi_{n-1}\iota_{n-1} = 0].
\end{aligned}$$

Then, d_n is given by $d_n([x]) = [\tilde{d}_n(x)]$, for each $[x] \in Q_n(X)$.

Step 3: we show that $d_n d_{n+1} = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
D_{n+1}(X) \arrow[r, hook, "\iota_{n+1}"] \arrow[d, "\tilde{d}_{n+1}|_{D_{n+1}(X)}"'] & \tilde{Q}_{n+1}(X) \arrow[r, "\pi_{n+1}"] \arrow[d, "\tilde{d}_{n+1}"] & \tilde{Q}_{n+1}(X)/D_{n+1}(X) \arrow[d, dashed, "\exists! d_{n+1}"] \\
D_n(X) \arrow[r, hook, "\iota_n"] \arrow[d, "\tilde{d}_n|_{D_n(X)}"'] & \tilde{Q}_n(X) \arrow[r, "\pi_n"] \arrow[d, "\tilde{d}_n"] & \tilde{Q}_n(X)/D_n(X) \arrow[d, dashed, "\exists! d_n"] \\
D_{n-1}(X) \arrow[r, hook, "\iota_{n-1}"] & \tilde{Q}_{n-1}(X) \arrow[r, "\pi_{n-1}"] & \tilde{Q}_{n-1}(X)/D_{n-1}(X)
\end{tikzcd}.$$

$$\begin{aligned}
d_n d_{n+1} = 0 &\iff d_n d_{n+1} \pi_{n+1} = 0 && [\pi_{n+1} \text{ is surjective}] \\
&\iff \pi_{n-1} \tilde{d}_n \tilde{d}_{n+1} = 0 && [\text{the diagram above commutes}] \\
&\iff \text{true} && [\text{by exercise 2 of sheet No. 9, } \tilde{d}_n \tilde{d}_{n+1} = 0]. \qquad \square
\end{aligned}$$

Exercise 10.2. Let $H_n^{\mathrm{cub}}(X) \coloneqq \ker d_n / \operatorname{im} d_{n+1}$ be the singular Homology with cubes as in exercise 10.1 and $b_n^{\mathrm{cub}}(X) \coloneqq \dim H_n^{\mathrm{cub}}(X)$. Show that for a point $\{p\}$ the Betti numbers are $b_0^{\mathrm{cub}}(\{p\}) = 1$ and $\forall n \geq 1\colon b_n^{\mathrm{cub}}(\{p\}) = 0$.

Solution. By exercise 2 of sheet No. 9, when X is a point we have $\tilde{d}_n = 0$ for all $n \geq 0$. This implies that $d_n = 0$ for all $n \geq 0$, because

$$\begin{aligned}
d_n = 0 &\iff d_n \pi_n = 0 && [\pi_n \text{ is surjective}] \\
&\iff \pi_{n-1} \tilde{d}_n = 0 && [\text{diagram defining } \tilde{d}_n \text{ commutes}] \\
&\iff \text{true}.
\end{aligned}$$

In other words, $\ker d_n = \tilde{Q}_n(\{p\})/D_0(\{p\})$ and $\operatorname{im} d_n = \{0\}$. By exercise 2 of sheet No. 9, $\dim \tilde{Q}_n(\{p\}) = 1$. It remains to compute $\dim D_n(\{p\})$. We claim that $D_n(\{p\}) = \tilde{Q}_n(\{p\})$ if $n \geq 1$ and that $D_0(\{p\}) = \{0\}$. If $n \geq 1$, then every $q\colon I^n \longrightarrow \{p\}$ is degenerate because q is constant. Therefore $D_n(\{p\}) = \tilde{Q}_n(\{p\})$ and $\dim D_n(\{p\}) = 1$ if $n \geq 1$. If $n = 0$, then the unique map $q\colon I^0 \longrightarrow \{p\}$ is not degenerate (the definition of degenerate is that $\exists i = 1, \ldots, n\colon q$ does not depend on x_i, but here $n = 0$ so such an i can't exist). So $D_0(\{p\}) = \{0\}$ and $\dim D_0(\{p\}) = 0$.

$$\begin{aligned}
b_n^{\mathrm{cub}}(\{p\}) &= \dim H_n^{\mathrm{cub}}(X) && [\text{by definition of } b_0^{\mathrm{cub}}] \\
&= \dim \frac{\ker d_n}{\operatorname{im} d_{n+1}} && [\text{by definition of } H_n^{\mathrm{cub}}] \\
&= \dim \ker d_n - \dim \operatorname{im} d_{n+1} && [\text{dimension of quotient of vector spaces}] \\
&= \dim \frac{\tilde{Q}_n(\{p\})}{D_n(\{p\})} - \dim\{0\} && [\text{above text}] \\
&= \dim \tilde{Q}_n(\{p\}) - \dim D_n(\{p\}) && [\text{dimension of quotient of vector spaces}] \\
&= \begin{cases} 1 - 0 & \text{if } n = 0 \\ 1 - 1 & \text{if } n \geq 1 \end{cases} && [\text{above text}] \\
&= \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}. \qquad \square
\end{aligned}$$

Exercise 10.3. Show that for every path-connected topological space $(X, \mathcal{O})$ we have that $b_0^{\text{cub}} = 1$.

Solution. Consider the following commutative diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, dashed] \\
D_1(X) \arrow[r, hook, "\iota_1"] \arrow[d, "\tilde{d}_1|_{D_1(X)}"'] & \tilde{Q}_1(X) \arrow[r, "\pi_1"] \arrow[d, "\tilde{d}_1"] & \tilde{Q}_1(X)/D_1(X) \arrow[d, dashed, "\exists! d_1"] \\
D_0(X) \arrow[r, hook, "\iota_0"] \arrow[d, "\tilde{d}_0|_{D_0(X)}"'] & \tilde{Q}_0(X) \arrow[r, "\pi_0"] \arrow[d, "\tilde{d}_0"] & \tilde{Q}_0(X)/D_0(X) \arrow[d, dashed, "\exists! d_0"] \\
\{0\} \arrow[r, hook, "\iota_{-1}"] & \{0\} \arrow[r, "\pi_{-1}"] & \{0\}
\end{tikzcd}.$$

Step 1: $D_0(X) = \{0\}$ and $\dim \operatorname{im} d_1 = \dim \operatorname{im} \tilde{d}_1$. By definition of degenerate cube, $D_0(X) = \{0\}$. Therefore, π_0 is an isomorphism, and

$$\begin{aligned}
\dim \operatorname{im} d_1 &= \dim \operatorname{im} d_1 \circ \pi_1 && [\pi_1 \text{ is surjective}] \\
&= \dim \operatorname{im} \pi_0 \circ \tilde{d}_1 && [\text{the diagram above commutes}] \\
&= \dim \operatorname{im} \tilde{d}_1 && [\pi_0 \text{ is an isomorphism}].
\end{aligned}$$

Step 2: $\dim \tilde{Q}_0(X)/\operatorname{im} \tilde{d}_1 = 1$. We start by showing that for every $\sigma, \delta \in C^0(\{0\}, X)$ (which is the set of generators of $\tilde{Q}_0(X)$) we have $\delta - \sigma \in \operatorname{im} \tilde{d}_1$. Define $p = \sigma(0)$ and $q = \delta(0)$. Since X is path connected, there exists $\gamma \colon [0,1] \longrightarrow X$ continuous such that $\gamma(0) = p$ and $\gamma(1) = q$. Then, $\gamma \in C^0(I, X)$ (which is the set of generators of $\tilde{Q}_1(X)$) and $\tilde{d}_1 \gamma = \delta - \sigma$. So, $\delta - \sigma \in \operatorname{im} \tilde{d}_1$. In other words, for every $\sigma, \delta \in C^0(\{0\}, X)$ we have $[\sigma] = [\delta] \in \tilde{Q}_0(X)/\operatorname{im} \tilde{d}_1$. Now, choose any $p \in X$ and define $\sigma \colon \{0\} \longrightarrow X, 0 \longmapsto p$. Then, $\sigma \in \tilde{Q}_0(X)$ and it has a corresponding equivalence class $[\sigma] \in \tilde{Q}_0(X)/\operatorname{im} \tilde{d}_1$. By the previous discussion, $\tilde{Q}_0(X)/\operatorname{im} \tilde{d}_1$ is generated by $[\sigma]$.

Step 3: putting everything together.

$$\begin{aligned}
&b_0^{\text{cub}}(\{p\}) \\
&\quad = \dim H_0^{\text{cub}}(X) && [\text{by definition of } b_0^{\text{cub}}] \\
&\quad = \dim \frac{\ker d_0}{\operatorname{im} d_1} && [\text{by definition of } H_0^{\text{cub}}] \\
&\quad = \dim \ker d_0 - \dim \operatorname{im} d_1 && [\text{dimension of quotient}] \\
&\quad = \dim \tilde{Q}_0(X)/D_0(X) - \dim \operatorname{im} d_1 && [d_0 \colon \tilde{Q}_0(X)/D_0(X) \longrightarrow \{0\} \\
& && \implies \ker d_0 = \tilde{Q}_0(X)/D_0(X)] \\
&\quad = \dim \tilde{Q}_0(X) - \dim D_0(X) - \dim \operatorname{im} d_1 && [\text{dimension of quotient}] \\
&\quad = \dim \tilde{Q}_0(X) - \dim \operatorname{im} d_1 && [\text{by step 1, } D_0 = \{0\}] \\
&\quad = \dim \tilde{Q}_0(X) - \dim \operatorname{im} \tilde{d}_1 && [\text{by step 1, } \dim \operatorname{im} \tilde{d}_1 = \dim \operatorname{im} d_1] \\
&\quad = \dim \tilde{Q}_0(X)/\operatorname{im} \tilde{d}_1 && [\text{dimension of quotient}] \\
&\quad = 1 && [\text{by step 2}].
\end{aligned}$$

$\square$

11 Exercise sheet No. 11

In this exercise sheet we go over an introduction to category theory and homological algebra. These are mathematical theories that can be used to talk about the various kinds of homologies in topological spaces. We then use these new concepts from category theory and homological algebra to solve exercise 4 (homotopy invariance of cubical homology) from lecture notes No. 10. Your goals are to:

- read the sketch of the solution of exercise 4;
- read onwards to learn about the new concepts from category theory and homological algebra;
- prove the lemmas in this text as they show up;
- in the end, use everything you learned to write the final step of the proof of homotopy invariance.

Exercise 11.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homotopy equivalent topological spaces. Show that $b_n^{\text{cub}}(X) = b_n^{\text{cub}}(X)$ for all $n \in \mathbb{N}_0$.

Solution sketch. We start with a proof sketch. As we saw before, it's possible to write the singular homology (cubical or with simplexes) in the language of category theory (definition 11.2 below) by saying it's a functor (definition 11.5 below). For cubical singular homology, we created a sequence of vector spaces $Q_n(X)$ together with maps $d_n \colon Q_n(X) \longrightarrow Q_{n-1}(X)$ (this data is what's called a chain complex (definition 11.12 below)). Given these vector spaces $Q_n(X)$ and d_n, we can define the homology (see definition 11.16) below of the chain complex $Q_n(X)$, which we are denoting by $H_n^{\text{cub}}(X)$. We will write down what we just said in terms of categories and functors: there are categories **Top** of topological spaces, **Vec** of vector spaces over $\mathbb{R}$, **Comp** of chain complexes. Also, there are functors $Q \colon \mathbf{Top} \longrightarrow \mathbf{Comp}$ (that to every topological space assigns its chain complex, that is, the family of Q_n's and d_n's - lemma 11.22) and $H_n \colon \mathbf{Comp} \longrightarrow \mathbf{Vec}$ (that to every chain complex assigns it's n-th homology vector space):

$$\mathbf{Top} \xrightarrow{Q} \mathbf{Comp} \xrightarrow{H_n} \mathbf{Vec}\ .$$

We saw this discussion already on exercise sheet No. 9 in the other case of homology with simplexes. Now, we are going to consider extra notions on the categories **Top** and **Comp**. Namely, in **Top** we have the notion of two maps of topological spaces being homotopic and in **Comp** we have the notion of two maps of chain complexes being chain homotopic (definition 11.14). In a way that we will make precise later, these notions are "equivalence relations" (we are going to call these special equivalence relations congruences - see definition 11.8 below) in our categories **Top** and **Comp**, and we can define new categories $\mathbf{Top}/\sim$, $\mathbf{Comp}/\sim$ which are the "quotients" (definition 11.9). We are also going to have "quotient maps" (definition 11.10) between the categories, which in this case are functors. So, we will have the following diagram:

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathbf{Comp} \arrow[r, "H_n"] \arrow[d, "\pi"] & \mathbf{Vec} \\
\mathbf{Top}/\sim & \mathbf{Comp}/\sim &
\end{tikzcd}.$$

Once we have all this machinery set up, the main steps to show that cubical homology is homotopy invariant are going to be showing that the functors Q and H_n descend to these quotient categories (lemmas 11.11, 11.19 and 11.23), so that we end up with a commutative diagram

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathbf{Comp} \arrow[r, "H_n"] \arrow[d, "\pi"] & \mathbf{Vec} \\
\mathbf{Top}/\sim \arrow[r, "Q"'] & \mathbf{Comp}/\sim \arrow[ur, "H_n"'] &
\end{tikzcd}.$$

Having this commutative diagram would finish the proof, because if X and Y are homotopy equivalent, then that means that in the category $\mathbf{Top}/\sim$ they are isomorphic, and then that means that $H_n^{\mathrm{cub}}(X) = H_nQ(X) \cong H_nQ(Y) = H_n^{\mathrm{cub}}(Y)$. Now we write everything we said precisely/with more detail. $\square$

11.2 Categories and functors

Definition 11.2. A **category** $\mathcal{C}$ is given by the following data

- A class $\mathcal{C}$, whose elements are the **objects** of the category;
- For each $A, B \in \mathcal{C}$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of **morphisms from** A **to** B. For $f \in \mathrm{Hom}(A, B)$, we write $f\colon A \longrightarrow B$;
- For each $A, B, C \in \mathcal{C}$, a **composition map**

$$\begin{aligned} \circ\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C) \\ (f, g) &\longmapsto gf \end{aligned}$$

satisfying the following axioms:

(domain and target) For each $A, B, C, D \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B) \cap \mathrm{Hom}_{\mathcal{C}}(C, D) = \varnothing$. If $f\colon A \longrightarrow B$, A is the **domain** of f and B is the **target** of f;

(identity morphism) For all $A \in \mathcal{C}$ there exists $\mathrm{id}_A\colon A \longrightarrow A$ such that for all $f\colon A \longrightarrow B$ we have that $\mathrm{id}_B f = f$ and $f\mathrm{id}_A = f$;

(associativity) If $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ and $h\colon C \longrightarrow D$ then $h(gf) = (hg)f$.

A quick note: a category $\mathcal{C}$ is the class of objects and the class of morphisms. In our definition, we wrote $\mathcal{C}$ for the class of objects as well. What we mean is that even though the category itself and it's class of objects are different things, we are committing an abuse of language and denoting the category and it's class of objects with the same symbol. Some examples of categories are:

Example 11.3.

Grp, where objects are groups and morphisms are group homomorphisms;

Top, where objects are topological spaces and morphisms are continuous maps;

Vec, where objects are vector spaces over $\mathbb{R}$ and morphisms are linear maps.

Definition 11.4. Let $\mathcal{C}$ be a category and $f\colon A \longrightarrow B$ be a morphism in $\mathcal{C}$. f is an **isomorphism** if there exists a morphism $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

A functor is going to be a "morphism" of categories. A category has objects and morphisms, so a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ should map objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and it should map morphisms of $\mathcal{C}$ to morphisms of $\mathcal{D}$. A functor must also preserve the other structure of the category, so it should preserve identities and composition.

Definition 11.5. Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** F **from** $\mathcal{C}$ **to** $\mathcal{D}$, denoted $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is given by the data

- a function $F\colon \mathcal{C} \longrightarrow \mathcal{D}$,
- for each $A, B \in \mathcal{C}$, a function $F\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FA, FB)$,

such that

(identity) $\forall A \in \mathcal{C}\colon F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$;

(composition) $\forall A, B, C \in \mathcal{C}\colon \forall f\colon A \longrightarrow B\colon \forall g\colon B \longrightarrow C\colon F(gf) = F(g)F(f)$.

Lemma 11.6. *Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor, and $f\colon A \longrightarrow B$ be an isomorphism in $\mathcal{C}$. Then, $F(f)\colon F(A) \longrightarrow F(B)$ is an isomorphism.*

Proof. By definition of isomorphism, we need to show that there exists a morphism $h\colon F(B) \longrightarrow F(A)$ such that $F(f)h = \mathrm{id}_{F(B)}$ and $hF(f) = \mathrm{id}_{F(A)}$. Since f is an isomorphism, there exists a morphism $g\colon B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. Then,

$$\begin{aligned} F(f)F(g) &= F(fg) && [F \text{ is a functor, so it preserves composition}] \\ &= F(\mathrm{id}_B) && [f \text{ and } g \text{ are inverses}] \\ &= \mathrm{id}_{F(B)} && [F \text{ is a functor, so it preserves identities}], \end{aligned}$$

and analogously $F(g)F(f) = \mathrm{id}_{F(A)}$. □

Lemma 11.7. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$, $G\colon \mathcal{D} \longrightarrow \mathcal{E}$ be functors. Then,*

$$\begin{aligned} FG\colon \quad \mathcal{C} &\longrightarrow \mathcal{E} \\ X &\longmapsto F(G(X)) \\ f \downarrow &\longmapsto \downarrow F(G(f)) \\ Y &\longmapsto F(G(Y)) \end{aligned}$$

is a functor.

Proof. FG preserves identities:

$$\begin{aligned} FG(\mathrm{id}_X) &= F(G(\mathrm{id}_X)) \\ &= F(\mathrm{id}_{G(X)}) \\ &= \mathrm{id}_{F(G(X))} \\ &= \mathrm{id}_{FG(X)}\,. \end{aligned}$$

FG preserves compositions:

$$\begin{aligned} FG(fg) &= F(G(fg)) \\ &= F(G(f)G(g)) \\ &= F(G(f))F(G(g)) \\ &= FG(f)FG(f). \end{aligned}$$

□

Definition 11.8. Let $\mathcal{C}$ be a category. A **congruence** on $\mathcal{C}$ is an assignment $\sim$ that for each ordered pair $\{A, B\}$ of objects of $\mathcal{C}$ gives an equivalence relation $\sim_{A,B}$ on $\mathrm{Hom}_{\mathcal{C}}(A, B)$, which satisfies the following property: for all $A, B, C \in \mathcal{C}$ and for all $f, f' \colon A \longrightarrow B$ and for all $g, g' \colon B \longrightarrow C$, if $f \sim_{A,B} f'$, $g \sim_{B,C} g'$, then $g \circ f \sim_{A,C} g' \circ f'$.

To make the notation simpler, we are going to omit the subscript in each equivalence relation and write only $\sim$ instead of $\sim_{A,B}$. But keep in mind that we have a family of equivalence relations, one for each ordered pair of objects in the category.

Definition 11.9. Let $\mathcal{C}$ be a category with a congruence $\sim$. Define the **quotient category** of $\mathcal{C}$, denoted $\mathcal{C}/\sim$, as follows.

(Objects) Objects in $\mathcal{C}$ are objects in $\mathcal{C}/\sim$;

(Morphisms) For $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}/\sim}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim$;

(Composition) For $[f] \colon A \longrightarrow B$, $[g] \colon B \longrightarrow C$ morphisms in $\mathcal{C}/\sim$, $[g] \circ [f] = [g \circ f]$.

By the definition of congruence, $\mathcal{C}/\sim$ is well defined and is a category. We denote by $\mathrm{id}_{A,\sim} = [\mathrm{id}_A]$ the identity morphism of the object A in $\mathcal{C}/\sim$.

Definition 11.10. Let $\mathcal{C}$ be a category with a congruence $\sim$, and let $\mathcal{C}/\sim$ be its quotient category. The **quotient functor** of $\mathcal{C}$ is the functor

$$\begin{aligned} \pi\colon\quad \mathcal{C} &\longrightarrow \mathcal{C}/\sim \\ A &\longmapsto A \\ f\downarrow &\longmapsto \downarrow[f] \\ B &\longmapsto B. \end{aligned}$$

It is routine to see that $\pi \colon \mathcal{C} \longrightarrow \mathcal{C}/\sim$ preserves identities and composition, so it is a functor.

Lemma 11.11. *Let $\mathcal{C}$ be a category with a congruence $\sim$, let $\mathcal{C}/\sim$ be the quotient category of $\mathcal{C}$, and let $\mathcal{A}$ be a category. If $F \colon \mathcal{C} \longrightarrow \mathcal{A}$ is a functor such that for all*

$A, B \in \mathcal{C}$ and for all morphisms $f, g\colon A \longrightarrow B$ in $\mathcal{C}$ we have $f \sim g \Longrightarrow F(f) = F(g)$, then there exists a unique functor F' such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{A} \\ \mathcal{C}/\sim \arrow[ur, "F'"'] & \end{tikzcd}.$$

Proof. Uniqueness: if F' is such a functor, then for every $A \in \mathcal{C}$ we have $F'(A) = F'(\pi(A)) = F(A)$ and for every $[f]\colon A \longrightarrow B$ a morphism in $\mathcal{C}/\sim$ (with representative $f\colon A \longrightarrow B$) we have $F'([f]) = F'(\pi(f)) = F(f)$. So, F' is uniquely determined.

Existence: define F' with the equations above. We need to check that the definition of F on morphisms is well posed. This is true by the property "$f \sim g \Longrightarrow F(f) = F(g)$". We also need to check that F' given this way is a functor:

$$\begin{aligned} F'([f][g]) &= F'([fg]) && [\text{def. composition in } \mathcal{C}/\sim] \\ &= F(fg) && [\text{def. } F'] \\ &= F(g)F(g) && [F \text{ is a functor}] \\ &= F'([f])F'([g]) && [\text{def. } F'], \end{aligned}$$

$$\begin{aligned} F'(\mathrm{id}_{A,\sim}) &= F'([\mathrm{id}_A]) && [[\mathrm{id}_A] \text{ is the identity of } A \text{ in } \mathcal{C}/\sim] \\ &= F(\mathrm{id}_A) && [\text{def. } F'] \\ &= \mathrm{id}_{F(A)} && [F \text{ is a functor}] \\ &= \mathrm{id}_{F'(A)} && [\text{def. } F']. \end{aligned}$$

□

11.3 Topological spaces

Again, with the definitions we just gave, there is a category **Top** whose objects are topological spaces, morphisms are continuous maps, and such that composition of morphisms is just the usual composition of functions. If we have two topological spaces X, Y and continuous maps $f, g\colon X \longrightarrow Y$, then we have a notion of the maps f, g being homotopic. We have already seen that "homotopic" is an equivalence relation and that if $f, f'\colon X \longrightarrow Y$ and $g, g'\colon Y \longrightarrow Z$ are two pairs of homotopic continuous maps, then $g \circ f$ and $g \circ f'$ are homotopic. Therefore, "homotopic" is a congruence on the category of topological spaces. Therefore, by lemma 11.11, we have a quotient category **Top**$/\sim$ and a quotient functor $\pi\colon$ **Top** $\longrightarrow$ **Top**$/\sim$.

11.4 Chain complexes

Definition 11.12. A **chain complex** of vector spaces over $\mathbb{R}$, C, is a sequence $\{C_n\}_{n\in\mathbb{Z}}$ of vector spaces over $\mathbb{R}$ and a sequence of linear maps $d_n\colon C_n \longrightarrow C_{n-1}$ such that $d_n d_{n+1} = 0$ for every $n \in \mathbb{Z}$ (in other words, such that $\operatorname{im} d_{n+1} \subset \ker d_n$).

Definition 11.13. Let C, D be chain complexes. A **chain map** from C to D is a sequence of linear maps $f_n \colon C_n \longrightarrow D_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_n \arrow[r, "d_n^D"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}.$$

The composition of two chain maps is again a chain map, so chain complexes of vector spaces over $\mathbb{R}$ and chain maps form a category which we are going to call **Comp**.

Definition 11.14. Let C, D be chain complexes and $f, g \colon C \longrightarrow D$ be chain maps. A **chain homotopy** from f to g is a sequence of linear maps $T_n \colon C_n \longrightarrow D_{n+1}$ such that $f_n - g_n = d_{n+1}^D T_n + T_{n-1} d_n^C$. In this case, we say that f is **chain homotopic** to g.

Lemma 11.15. *"Chain homotopic" is a congruence on the category of chain complexes of vector spaces over* $\mathbb{R}$*.*

Proof. We show that "chain homotopic" is an equivalence relation.

Reflexive: if f is a chain map, then $T_n = 0$ is a chain homotopy from f to f.

Symmetric: if T_n is a chain homotopy from f to g, then $-T_n$ is a chain homotopy from g to f.

Transitive: If T_n is a chain homotopy from f to g and U_n is a chain homotopy from g to h, then $T_n + U_n$ is a chain homotopy from f to h:

$$\begin{aligned}
f_n - h_n &= f_n - g_n + g_n - h_n \\
&= d_{n+1}^D T_n + T_{n-1} d_n^C + d_{n+1}^D U_n + U_{n-1} d_n^C \\
&= d_{n+1}^D (T_n + U_n) + (T_{n-1} + U_{n-1}) d_n^C.
\end{aligned}$$

We show that in addition to being an equivalence relation, "chain homotopic" is a congruence. For this, assume that f, f' and g, g' are chain maps and that T_n is a chain homotopy from f to f' and that U_n is a chain homotopy from g to g'. Then, $g_{n+1} T_n$ is a chain homotopy from gf to gf':

$$\begin{aligned}
g_n f_n - g_n f_n' &= g_n (f_n - f_n') \\
&= g_n (d_{n+1}^D T_n + T_{n-1} d_n^C) \\
&= g_n d_{n+1}^D T_n + g_n T_{n-1} d_n^C \\
&= d_{n+1}^E g_{n+1} T_n + g_n T_{n-1} d_n^C,
\end{aligned}$$

and by an analogous computation $U_n f_n'$ is a chain homotopy from gf' to $g'f'$. By transitivity, gf is chain homotopic to $g'f'$. $\square$

By lemmas 11.11 and 11.15, we have a quotient category **Cong**$/\sim$ and a quotient functor $\pi \colon$ **Cong** $\longrightarrow$ **Cong**$/\sim$.

11.5 Homology

Definition 11.16. Let C be a chain complex of vector spaces over $\mathbb{R}$. The n-th **Homology** vector space of C is $H_n(C) \coloneqq \frac{\ker d_n}{\operatorname{im} d_{n+1}}$.

Lemma 11.17. *Let C, D be chain complexes of vector spaces over $\mathbb{R}$ and $f\colon C \longrightarrow D$ be a chain map. Consider $f_n\colon C_n \longrightarrow D_n$. Then, f_n maps $\operatorname{im} d^C_{n+1}$ to $\operatorname{im} d^D_{n+1}$ and it maps $\ker d^C_n$ to $\ker d^D_n$. Also, there exists a unique $H_n(f)$ making the following diagram commute:*

$$\begin{tikzcd}
\operatorname{im} d^C_{n+1} \arrow[r, hook, "\iota^C_n"] \arrow[d, "f_n"'] & \ker d^C_n \arrow[r, "\pi^C_n"] \arrow[d, "f_n"] & \ker d^C_n / \operatorname{im} d^C_{n+1} \arrow[d, "H_n(f)"] \\
\operatorname{im} d^D_{n+1} \arrow[r, hook, "\iota^D_n"] & \ker d^D_n \arrow[r, "\pi^D_n"] & \ker d^D_n / \operatorname{im} d^D_{n+1}
\end{tikzcd},$$

*which is given by $H_n(f)([x]) = [f_n(x)]$ for each $[x] \in \ker d^C_n / \operatorname{im} d^C_{n+1}$. We say that $H_n(f)\colon H_n(C) \longrightarrow H_n(D)$ is the **map induced by H_n and f**.*

Proof. f_n maps $\operatorname{im} d^C_{n+1}$ to $\operatorname{im} d^D_{n+1}$: it suffices to assume that $x \in \operatorname{im} d^C_{n+1}$ and to prove that $f_n(x) \in \operatorname{im} d^D_{n+1}$. Since $x \in \operatorname{im} d^C_{n+1}$, there exists an $a \in C_{n+1}$ such that $d^C_{n+1}(a) = x$.

$$\begin{aligned}
f_n(x) &= f_n d^C_{n+1}(a) && [\text{def. } a] \\
&= d^D_{n+1} f_{n+1}(a) && [f \text{ is a chain map}] \\
&\in \operatorname{im} d^D_{n+1} && [\text{def. image}].
\end{aligned}$$

f_n maps $\ker d^C_n$ to $\ker d^D_n$: it suffices to assume that $x \in \ker d^C_n$ and to prove that $f_n(x) \in \ker d^D_n$.

$$\begin{aligned}
d^D_n f_n(x) &= f_{n-1} d^C_n(x) && [f \text{ is a chain map}] \\
&= f_{n-1} 0 && [x \in \ker d^C_n] \\
&= 0.
\end{aligned}$$

There exists a unique $H_n(f)$ such that the diagram above commutes: by the universal property of the quotient, it suffices to assume that $x \in \ker d^C_n$ is such that $\pi^C_n(x) = 0$, and to prove that $\pi^D_n f_n(x) = 0$. Note that $x \in \ker \pi^C_n = \operatorname{im} d^C_{n+1}$, because π^C_n is the quotient map.

$$\begin{aligned}
\pi^D_n f_n(x) &= \pi^D_n f_n \iota^C_n(x) && [x \in \operatorname{im} d^C_{n+1}] \\
&= \pi^D_n \iota^D_n f_n(x) && [\text{the left square commutes}] \\
&= 0 && [\pi^D_n \iota^D_n = 0].
\end{aligned}$$

For $x \in \ker d^C_n$, $H_n(f)$ is given by $H_n(f)([x]) = [f_n(x)]$, again by the universal property of the quotient. ☐

Lemma 11.18. *The n-th Homology vector space*

$$\begin{aligned}
H_n\colon \mathbf{Comp} &\longrightarrow \mathbf{Vec} \\
C &\longmapsto H_n(C) \\
f \downarrow &\longmapsto \downarrow H_n(f) \\
D &\longmapsto H_n(D)
\end{aligned}$$

is a functor.

Proof. H_n preserves identities: it suffices to assume $[x] \in H_n(C) = \frac{\ker d_n^C}{\operatorname{im} d_{n+1}^C}$ with representative $x \in \ker d_n^C$ and to prove that $H_n(\mathrm{id}_C)([x]) = \mathrm{id}_{H_n(C)}([x])$.

$$\begin{aligned} H_n(\mathrm{id}_C)([x]) &= [(\mathrm{id}_C)_n(x)] && \text{[def. } H_n \text{ on morphisms]} \\ &= [\mathrm{id}_{C_n}(x)] && \text{[identities on the category of chain complexes]} \\ &= [x] \\ &= \mathrm{id}_{H_n(C)}([x]). \end{aligned}$$

H_n preserves compositions: let $f\colon C \longrightarrow D$ and $g\colon D \longrightarrow E$ be morphisms of chain complexes. It suffices to assume $[x] \in H_n(C) = \frac{\ker d_n^C}{\operatorname{im} d_{n+1}^C}$ with representative $x \in \ker d_n^C$ and to prove that $H_n(g)H_n(f)([x]) = H_n(gf)([x])$.

$$\begin{aligned} H_n(g)H_n(f)([x]) &= H_n(g)([f_n(x)]) && \text{[def. } H_n \text{ on morphisms]} \\ &= [g_n f_n(x)] && \text{[def. } H_n \text{ on morphisms]} \\ &= [(gf)_n(x)] && \text{[compositions on the cat. of chain complexes]} \\ &= H_n(gf)([x]) && \text{[def. } H_n \text{ on morphisms].} \end{aligned}$$

□

Lemma 11.19. *Let C, D be chain complexes and $f, g\colon C \longrightarrow D$ be chain maps. If f, g are chain homotopic, then $H_n(f) = H_n(g)\colon H_n(C) \longrightarrow H_n(D)$.*

Proof. Let $[v] \in H_n(C)$, with representative $v \in \ker d_n^C$. We must show that $H_n(f)([v]) = H_n(g)([v])$. Let T_n be a chain homotopy from f to g. Then,

$$\begin{aligned} &H_n(f)([v]) \\ &\quad = [f_n(v)] && \text{[def. } H_n(f)] \\ &\quad = [g_n(v)] + [d_{n+1}^D T_n(v)] + [T_{n-1} d_n^C(v)] && [T_n \text{ is a chain homotopy from } f \text{ to } g] \\ &\quad = [g_n(v)] + [d_{n+1}^D T_n(v)] && [v \in \ker d_n^C] \\ &\quad = [g_n(v)] && [d_{n+1}^D T_n(v) \in \operatorname{im} d_{n+1}^D] \\ &\quad = H_n(g)([v]) && \text{[def. } H_n(g)]. \end{aligned}$$

□

By lemmas 11.11 and 11.19, the n-th Homology functor descends to a functor on the quotient category, which we denote also by H_n:

$$\begin{tikzcd} \mathbf{Comp} \arrow[r, "H_n"] \arrow[d, "\pi"'] & \mathbf{Vec} \\ \mathbf{Comp}/\sim \arrow[ur, "H_n"'] & \end{tikzcd}.$$

11.6 Cubical singular chain complex

If we have a topological space X, we have already seen that we can form it's cubical singular chain complex, $Q_n(X)$ which has differentials $d_n\colon Q_n(X) \longrightarrow Q_{n-1}(X)$. We denote by $Q(X)$ the chain complex, so $Q(X)$ is the data $\{Q_n(X)\}_{n\in\mathbb{Z}}$, $\{d_n\}_{n\in\mathbb{Z}}$.

Definition 11.20. Let X, Y be topological spaces. Let $f\colon X \longrightarrow Y$ be a continuous map. Define the **chain map induced by** f, denoted $Q(f)\colon Q(X) \longrightarrow Q(Y)$, as follows.

$Q(f)$ is going to be a chain map, so we need to say what is $Q_n(f)\colon Q_n(X) \longrightarrow Q_n(Y)$ for each n. We start by defining a linear map $\tilde{Q}_n(f)\colon \tilde{Q}_n(X) \longrightarrow \tilde{Q}_n(Y)$. By linearity, it suffices to say what is $\tilde{Q}_n(f)(\sigma)$ for each $\sigma \in C^0(I^n, X)$. We define $\tilde{Q}_n(f)(\sigma) = f \circ \sigma \in C^0(I^n, Y) \subset \tilde{Q}_n(Y)$. Then, we can check that $\tilde{Q}_n(f)$ maps $D_n(X)$ to $D_n(Y)$. By the universal property of the quotient, there exists a unique linear map $Q_n(f)$ such that the diagram

$$\begin{tikzcd}
D_n(X) \arrow[r, hook, "\iota_n^X"] \arrow[d, "\tilde{Q}_n(f)|_{D_n(X)}"'] & \tilde{Q}_n(X) \arrow[r, "\pi_n^X"] \arrow[d, "\tilde{Q}_n(f)"] & Q_n(X) \arrow[d, dashed, "\exists! Q_n(f)"] \\
D_n(Y) \arrow[r, hook, "\iota_n^Y"] & \tilde{Q}_n(Y) \arrow[r, "\pi_n^Y"] & Q_n(Y)
\end{tikzcd}$$

commutes, which is given by $Q_n(f)([x]) = [\tilde{Q}_n(f)(x)]$ for every $[x] \in Q_n(X)$.

Lemma 11.21. *In definition 11.20, $Q(f)\colon Q(X) \longrightarrow Q(Y)$ is a map of chain complexes.*

Proof. We want to prove that $d_n^Y Q_n(f) = Q_{n-1}(f) d_n^X$. Since π_n^X is surjective, it suffices to show that $d_n^Y Q_n(f) \pi_n^X = Q_{n-1}(f) d_n^X \pi_n^X$. By linearity, it suffices to assume that $\sigma \in C^0(I^n, X)$ and to prove that $d_n^Y Q_n(f) \pi_n^X(\sigma) = Q_{n-1}(f) d_n^X \pi_n^X(\sigma)$.

$$\begin{aligned}
d_n^Y Q_n(f) \pi_n^X(\sigma) &= d_n^Y Q_n(f)([\sigma]) && [\text{def. } \pi_n] \\
&= d_n^Y\Big([\tilde{Q}_n(f)(\sigma)]\Big) && [\text{def. } Q_n(f)] \\
&= d_n^Y([f \circ \sigma]) && [\text{def. } \tilde{Q}_n(f)] \\
&= [\tilde{d}_n^Y(f \circ \sigma)] && [\text{def. } d_n^Y] \\
&= \left[\sum_{j=1}^{n} (-1)^j \Big(f \circ \sigma \circ I_{j,0}^n - f \circ \sigma \circ I_{j,1}^n\Big)\right] && [\text{def. } \tilde{d}_n^Y] \\
&= \left[\tilde{Q}_{n-1}(f) \sum_{j=1}^{n} (-1)^j \Big(\sigma \circ I_{j,0}^n - \sigma \circ I_{j,1}^n\Big)\right] && [\text{def. } \tilde{Q}_{n-1}(f)] \\
&= Q_{n-1}(f) \left[\sum_{j=1}^{n} (-1)^j \Big(\sigma \circ I_{j,0}^n - \sigma \circ I_{j,1}^n\Big)\right] && [\text{def. } Q_{n-1}(f)] \\
&= Q_{n-1}(f)\Big([\tilde{d}_n^X(\sigma)]\Big) && [\text{def. } \tilde{d}_n^X] \\
&= Q_{n-1}(f) d_n^X([\sigma]) && [\text{def. } d_n^X] \\
&= Q_{n-1}(f) d_n^X \pi_n(\sigma) && [\text{def. } \pi_n^X].
\end{aligned}$$
$\square$

Lemma 11.22. *The cubical singular chain complex*

$$\begin{aligned}
Q\colon \mathbf{Top} &\longrightarrow \mathbf{Comp} \\
X &\longmapsto Q(X) \\
f \downarrow &\longmapsto \downarrow Q(f) \\
Y &\longmapsto Q(Y)
\end{aligned}$$

is a functor.

Proof. Q preserves identities: we have to show that $Q(\mathrm{id}_X) = \mathrm{id}_{Q(X)}$, i.e. that $Q_n(\mathrm{id}_X) = \mathrm{id}_{Q_n(X)}$. Since π_n^X is surjective, it suffices to show that $Q_n(\mathrm{id}_X)\pi_n^X = \mathrm{id}_{Q_n(X)}\pi_n^X$. By linearity, it suffices to assume that $\sigma \in C^0(I^n, X)$ and to prove that $Q_n(\mathrm{id}_X)\pi_n^X(\sigma) = \mathrm{id}_{Q_n(X)}\,\pi_n^X(\sigma)$.

$$\begin{aligned}
Q_n(\mathrm{id}_X)\pi_n^X(\sigma) &= [\tilde{Q}_n(\mathrm{id}_X)(\sigma)] \\
&= [\mathrm{id}_X\,\sigma] \\
&= [\sigma] \\
&= \mathrm{id}_{Q_n(X)}([\sigma]) \\
&= \mathrm{id}_{Q_n(X)}\,\pi_n^X(\sigma).
\end{aligned}$$

Q preserves compositions: we have to show that $Q(g)Q(f) = Q(gf)$, i.e. that $Q_n(g)Q_n(f) = Q_n(gf)$. Since π_n^X is surjective, it suffices to show that $Q_n(g)Q_n(f)\pi_n^X = Q_n(gf)\pi_n^X$. By linearity, it suffices to assume that $\sigma \in C^0(I^n, X)$ and to prove that $Q_n(g)Q_n(f)\pi_n^X(\sigma) = Q_n(gf)\pi_n^X(\sigma)$.

$$\begin{aligned}
Q_n(g)Q_n(f)\pi_n^X(\sigma) &= Q_n(g)Q_n(f)([\sigma]) \\
&= Q_n(g)([\tilde{Q}_n(f)(\sigma)]) \\
&= [\tilde{Q}_n(g)\tilde{Q}_n(f)(\sigma)] \\
&= [\tilde{Q}_n(g)(f \circ \sigma)] \\
&= [g \circ f \circ \sigma] \\
&= [\tilde{Q}_n(gf)(\sigma)] \\
&= Q_n(gf)([\sigma]) \\
&= Q_n(gf)\pi_n^X(\sigma).
\end{aligned}$$

□

As we said before, what we did so far is setting up some categorical language for this problem. The following lemma is the main step of the solution of our problem.

Lemma 11.23. *Let X, Y be topological spaces, and $f, g\colon X \longrightarrow Y$ be continuous maps. If $f, g\colon X \longrightarrow Y$ are homotopic then $Q(f), Q(g)\colon Q(X) \longrightarrow Q(Y)$ are chain homotopic.*

Proof. Let H be a homotopy from f to g. We wish to show that there exists a chain homotopy $P_n\colon Q_n(X) \longrightarrow Q_{n+1}(Y)$ from $Q(f)$ to $Q(g)$, i.e. that $Q_n(f) - Q_n(g) = d_{n+1}^Y P_n + P_{n-1} d_n^X$. We start by defining a map $\tilde{P}_n\colon \tilde{Q}_n(X) \longrightarrow \tilde{Q}_{n+1}(Y)$. By linearity, it suffices to say what is $\tilde{P}_n\sigma$ for each $\sigma \in C^0(I^n, X)$. Note that

$$I^{n+1} = I^n \times I \xrightarrow{\sigma \times \mathrm{id}_I} X \times I \xrightarrow{H} Y.$$

Then, we define $\tilde{P}_n(\sigma) = (-1)^{n+1} H \circ (\sigma \times \mathrm{id}_I)$. Then, $\tilde{P}_n$ maps $D_n(X)$ to $D_{n+1}(Y)$. Therefore, by the universal property of the quotient, there exists a unique map $P_n\colon Q_n(X) \longrightarrow Q_{n+1}(Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
D_n(X) \arrow[r, hook, "\iota_n^X"] \arrow[d, "\tilde{P}_n|_{D_n(X)}"'] & \tilde{Q}_n(X) \arrow[r, "\pi_n^X"] \arrow[d, "\tilde{P}_n"] & Q_n(X) \arrow[d, dashed, "\exists! P_n"] \\
D_{n+1}(Y) \arrow[r, hook, "\iota_{n+1}^Y"] & \tilde{Q}_{n+1}(Y) \arrow[r, "\pi_{n+1}^Y"] & Q_{n+1}(Y)
\end{tikzcd}.$$

For each $[x] \in Q_n(X)$, $P_n([x]) = [\tilde{P}_n(x)]$. We claim that P_n is the desired chain homotopy. We start by showing that for $k = \{0, 1\}$, $H \circ (\sigma \times \mathrm{id}_I) \circ I^{n+1}_{j,k} = H \circ ((\sigma \circ I^n_{j,0}) \times \mathrm{id}_I)$.

$$\begin{aligned} H \circ (\sigma \times \mathrm{id}_I) \circ I^{n+1}_{j,k}(x_1, \ldots, x_n) &= H \circ (\sigma \times \mathrm{id}_I)(x_1, \ldots, x_{j-1}, k, x_j, \ldots, x_n) \\ &= H(\sigma(x_1, \ldots, x_{j-1}, k, x_j, \ldots, x_{n-1}), x_n) \\ &= H(\sigma \circ I^n_{j,k}(x_1, \ldots, x_{n-1}), x_n) \\ &= H \circ ((\sigma \circ I^n_{j,0}) \times \mathrm{id}_I)(x_1, \ldots, x_n). \end{aligned}$$

We now show that $Q_n(f) - Q_n(g) = d^Y_{n+1} P_n + P_{n-1} d^X_n$. Since $\pi^X_n : \tilde{Q}_n(X) \longrightarrow Q_n(X)$ is surjective, it suffices to show that $Q_n(f)\pi^X_n - Q_n(g)\pi^X_n = d^Y_{n+1} P_n \pi^X_n + P_{n-1} d^X_n \pi^X_n$. By linearity, it suffices to assume that $\sigma \in C^0(I^n, X)$ and to prove that $Q_n(f)\pi^X_n(\sigma) - Q_n(g)\pi^X_n(\sigma) = d^Y_{n+1} P_n \pi^X_n(\sigma) + P_{n-1} d^X_n \pi^X_n(\sigma)$.

$$\begin{aligned} & d^Y_{n+1} P_n \pi^X_n(\sigma) + P_{n-1} d^X_n \pi^X_n(\sigma) \\ &\quad = d^Y_{n+1} P_n([\sigma]) + P_{n-1} d^X_n([\sigma]) \\ &\quad = d^Y_{n+1}([\tilde{P}_n(\sigma)]) + P_{n-1}([\tilde{d}^X_n(\sigma)]) \\ &\quad = [\tilde{d}^Y_{n+1} \tilde{P}_n(\sigma)] + [\tilde{P}_{n-1} \tilde{d}^X_n(\sigma)] \\ &\quad = [\tilde{d}^Y_{n+1} \tilde{P}_n(\sigma) + \tilde{P}_{n-1} \tilde{d}^X_n(\sigma)] \\ &\quad = [(-1)^{n+1} \tilde{d}^Y_{n+1}(H \circ (\sigma \times \mathrm{id}_I)) + (-1)^n \tilde{P}_{n-1}(\tilde{d}^X_n \sigma)] \\ &\quad = \Big[(-1)^{n+1} \sum_{j=1}^{n+1} (-1)^j \Big(H \circ (\sigma \times \mathrm{id}_I) \circ I^{n+1}_{j,0} - H \circ (\sigma \times \mathrm{id}_I) \circ I^{n+1}_{j,1}\Big) \\ &\qquad + (-1)^n \sum_{j=1}^{n} (-1)^j \Big(H \circ ((\sigma \circ I^n_{j,0}) \times \mathrm{id}_I) - H \circ ((\sigma \circ I^n_{j,k}) \times \mathrm{id}_I)\Big)\Big] \\ &\quad = \Big[(-1)^{n+1}(-1)^{n+1}\Big(H \circ (\sigma \times \mathrm{id}_I) \circ I^{n+1}_{n+1,0} - H \circ (\sigma \times \mathrm{id}_I) \circ I^{n+1}_{n+1,1}\Big)\Big] \\ &\quad = [H(\cdot, 0) \circ \sigma - H(\cdot, 1) \circ \sigma] \\ &\quad = [f \circ \sigma] - [g \circ \sigma] \\ &\quad = [\tilde{Q}_n(f)(\sigma)] - [\tilde{Q}_n(g)(\sigma)] \\ &\quad = Q_n(f)([\sigma]) - Q_n(g)([\sigma]) \\ &\quad = Q_n(f)\pi^X_n(\sigma) - Q_n(g)\pi^X_n(\sigma). \end{aligned}$$

□

By lemmas 11.11 and 11.23, the cubical chain complex functor descends to a functor on the quotient categories, which we denote also by Q:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathbf{Comp} \arrow[d, "\pi"] \\ \mathbf{Top}/\sim \arrow[r, "Q"'] & \mathbf{Comp}/\sim \end{tikzcd} .$$

11.7 Conclusion

By everything we proved, we have a commutative diagram of categories and functors

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathbf{Comp} \arrow[r, "H_n"] \arrow[d, "\pi"] & \mathbf{Vec} \\
\mathbf{Top}/\sim \arrow[r, "Q"'] & \mathbf{Comp}/\sim \arrow[ur, "H_n"'] &
\end{tikzcd},$$

so now we can finish the proof.

Exercise 11.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be homotopy equivalent topological spaces. Show that $b_n^{\text{cub}}(X) = b_n^{\text{cub}}(X)$ for all $n \in \mathbb{N}_0$.

Solution. We start by showing that as objects of the category $\mathbf{Top}/\sim$, X and Y are isomorphic. Since X and Y are homotopy equivalent, there exist $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$ (here $\sim$ denotes "homotopic to"). Then, consider the morphisms $[f]\colon X \longrightarrow Y$ and $[g]\colon Y \longrightarrow X$ in $\mathbf{Top}/\sim$. Then,

$$\begin{aligned}
[f][g] &= [fg] && [\text{def. of composition in } \mathbf{Top}/\sim] \\
&= [\mathrm{id}_B] && [fg \sim \mathrm{id}_B] \\
&= \mathrm{id}_{B,\sim},
\end{aligned}$$

and $[g][f] = [\mathrm{id}_A] =$ identity of A in $\mathbf{Top}/\sim$. So X and Y are isomorphic in $\mathbf{Top}/\sim$. Then,

$$\begin{aligned}
b_n^{\text{cub}}(X) &= \dim H_n^{\text{cub}}(X) && [\text{def. } b_n^{\text{cub}}] \\
&= \dim H_n Q(X) && [\text{def. } H_n^{\text{cub}}] \\
&= \dim H_n Q(Y) && [X \text{ and } Y \text{ are isomorphic in } \mathbf{Top}/\sim\text{, lemma 11.6}] \\
&= \dim H_n^{\text{cub}}(Y) && [\text{def. } H_n^{\text{cub}}] \\
&= b_n^{\text{cub}}(Y) && [\text{def. } b_n^{\text{cub}}].
\end{aligned}$$

□

12 Exercise sheet No. 12

Exercise 12.1. Write down B_1 and B_2 explicitly.

Solution. Recall first the definitions for every n:

$$B_0 = id_{\Delta^0} : \Delta^0 \to \Delta^0,$$

and

$$B_n = \beta^n_{n-1}\left(B^{\Delta^n}_{n-1}(d_n id_{\Delta^n})\right), \text{ if } n > 0,$$

where

$$\beta^n_k : C_k(\Delta^n) \to C_{k+1}(\Delta^n),$$

$$\beta^n_k(\phi)\left(\sum_{i=1}^{k+2} t_i e_i\right) = \begin{cases} t_1\mu_n + (1-t_1)\phi\left(\frac{1}{1-t_1}\sum_{i=1}^{k+1} t_{i+1}e_i\right) & t_1 \neq 1 \\ \mu_n & t_1 = 1, \end{cases}$$

for any continuous map $\phi : \Delta^k \to \Delta^n$ and with $\mu_n = \frac{1}{n+1}(1,\dots,1)$ the barycenter of Δ^n, and

$$B^X_n : C_n(X) \to C_n(X)$$
$$B^X_n(\psi) = \psi_\# B_n,$$

for any space X and any continuous map $\psi : \Delta^n \to X$. We now unwrap the definition. For $n = 1$ we have:

$$\begin{aligned} B_1 =&\beta^1_0(B^{\Delta^1}_0(d_1 id_{\Delta^1})) \\ =&\beta^1_0(B^{\Delta^1}_0(e_2 - e_1)) \\ =&\beta^1_0((e_2)_\# B_0 - (e_1)_\# B_0) \\ =&\beta^1_0(e_2 \circ id_{\Delta^0} - e_1 \circ id_{\Delta^0}) \\ =&\beta^1_0(e_2) - \beta^1_0(e_1) \end{aligned} \tag{5}$$

Note that the above minus sign is *formal*. Evaluating at a point of Δ^1, we have

$$\beta^1_0(e_i)\left(\sum_{i=1}^{2} t_i e_i\right) = t_1\mu_1 + (1-t_1)e_i,$$

for $i = 1, 2$ (where we have used that e_i is a constant map and therefore there is no need to distinguish between cases). See Figure 2.

We now do $n = 2$. Denote $f_j := \partial^j_2 id_{\Delta^2} : \Delta^1 \to \Delta^2$ the j-th face of the 2-simplex, which one may parametrize as $f_j\,(te_1 + (1-t)e_2) = e_j + t(e_{j+1} - e_j)$ (where we take the index j mod 3). Then

$$\begin{aligned} B_2 =&\beta^2_1(B^{\Delta^2}_1(d_2 id_{\Delta^2})) \\ =&\beta^2_1(B^{\Delta^2}_1(f_1 - f_2 + f_3)) \\ =&\beta^2_1((f_1 - f_2 + f_3)_\# B_1) \\ =&\beta^2_1((f_1 - f_2 + f_3)_\#(\beta^1_0(e_2) - \beta^1_0(e_1)) \\ =&\sum_{j=1}^{3}\sum_{i=1}^{2}(-1)^{i+j+1}\Delta_{ij}, \end{aligned} \tag{6}$$

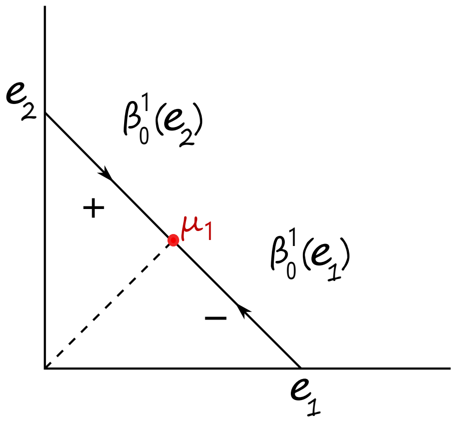

Figure 2: The barycentric subdivision of Δ^1. Roughly speaking, it is subdivided into two smaller 1-simplices with opposite signs.

where $\Delta_{ij} = \beta_1^2(f_j \circ \beta_0^1(e_i))$. Again, note that the expression for B_2 is just a *formal* alternated sum. To understand each term, we evaluate at a point of Δ^2, and get

$$\Delta_{ij}\left(\sum_{i=1}^{3} t_i e_i\right) = \begin{cases} t_1\mu_2 + (1-t_1)f_j\left(\frac{t_2}{1-t_1}\mu_1 + (1-\frac{t_2}{1-t_1})e_i\right) & t_1 \neq 1 \\ \mu_2 & t_1 = 1. \end{cases}$$

Using $\mu_1 = \frac{1}{2}(e_1 + e_2)$, we can write

$$\frac{t_2}{1-t_1}\mu_1 + \left(1 - \frac{t_2}{1-t_1}\right)e_i = \left(1 - \frac{t_2}{2(1-t_1)}\right)e_i + \frac{t_2}{2(1-t_1)}e_{i+1},$$

where we take the index i mod 2, and therefore

$$f_j\left(\frac{t_2}{1-t_1}\mu_1 + \left(1 - \frac{t_2}{1-t_1}\right)e_i\right) = e_j + t(i)(e_{j+1} - e_j),$$

where $t(i) = \begin{cases} 1 - \frac{t_2}{2(1-t_1)}, & i = 1 \\ \frac{t_2}{2(1-t_1)} & i = 2. \end{cases}$

We then get

$$\Delta_{ij}\left(\sum_{i=1}^{3} t_i e_i\right) = t_1\mu_2 + (1-t_1)(e_j + t(i)(e_{j+1} - e_j)),$$

with no need to distinguish between cases by noting that the above expression is well-defined for $t_1 = 1$. See Figure 3.

□

Exercise 12.2. Show that $\phi_\# T_{k-1}^{\Delta^k} = T_{k-1}^{\Delta^n}\phi_\#$, for every $\phi : \Delta^k \to \Delta^n$.

Solution. Recall the definitions:

$$T_0 = \beta_0^0(id_{\Delta^0}) \in C_1(\Delta^0),$$

and

$$T_n = \beta_n^n\left(id_{\Delta^n} - T_{n-1}^{\Delta^n} d_n id_{\Delta^n}\right) \in C_{n+1}(\Delta^n), \text{ if } n > 0.$$

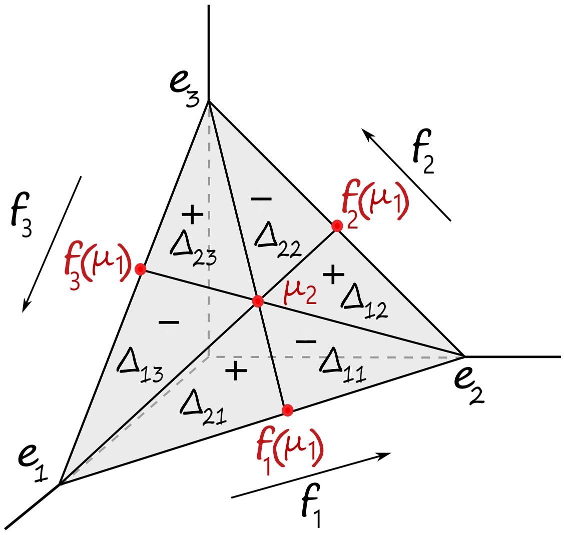

Figure 3: The barycentric subdivision of Δ^2, which decomposes it into a formal alternated sum of smaller copies of itself.

where

$$T^X_{n-1} : C_{n-1}(X) \to C_n(X)$$
$$T^X_{n-1}\psi = \psi_\# T_{n-1},$$

for $\psi : \Delta^{n-1} \to X$ and for any space X. The diagram we wish to show to be commutative is then:

$$\begin{array}{ccc} C_{k-1}(\Delta^k) & \xrightarrow{\phi_\#} & C_{k-1}(\Delta^n) \\ {\scriptstyle T^{\Delta^k}_{k-1}}\downarrow & & \downarrow{\scriptstyle T^{\Delta^n}_{k-1}} \\ C_k(\Delta^k) & \xrightarrow[\phi_\#]{} & C_k(\Delta^n) \end{array} .$$

This fact is actually independent on the definition of T_n, and follows simply from the covariance of the push-forward functor. Indeed, for arbitrary ψ we have:

$$\begin{aligned} \phi_\# T^{\Delta^k}_{k-1}\psi &= \phi_\#\psi_\# T_{k-1} \\ &= (\phi\circ\psi)_\# T_{k-1} \\ &= T^{\Delta^n}_{k-1}(\phi\circ\psi) \\ &= T^{\Delta^n}_{k-1}\phi_\#(\psi). \end{aligned} \tag{7}$$

□

Exercise 12.3. Show that the equation

$$d_{n+1}T^X_n + T^X_{n-1}d_n = id_{C_n(X)} - B^X_n$$

implies that the induced map in homology

$$\overline{B}^X_n : H_n(X) \to H_n(X)$$

is the identity $id_{H_n(X)}$.

Solution. Let us first remark that this is a general fact: The above equation, by definition, means that B_n^X is homotopic to the identity or nullhomotopic (as a chain map), where the homotopy is provided by the maps T_n^X. The following proof can be stated as: *nullhomotopic chain maps induce the identity in homology.* Moreover, a further general fact is that nullhomotopic maps of spaces induce nullhomotopic chain maps of the singular homology complexes.

The proof is straightforward: if $[c] \in H_n(X)$ with $c \in C_n(X)$ such that $d_n c = 0$, then the above homotopy equation implies

$$d_{n+1} T_n^X(c) = c - B_n^X(c),$$

therefore the class of $c - B_n^X(c)$ is 0, and therefore $id_{H_n(X)}[c] = [c] = [B_n^X(c)] = \overline{B}_n^X([c]) \in H_n(X)$. $\square$

13 Exercise sheet No. 13

Exercise 13.1. Show that it's possible to write the two-dimensional torus $T^2 = S^1 \times S^1$ as $T^2 = U \cup V$, where $U, V \subset T^2$ are open subsets with the property that U and V are both homotopy equivalent to a circle and $U \cap V$ is homotopy equivalent to the disjoint union of two circles $S^1 \sqcup S^1$.

Solution. Recall that $S^1 \coloneqq \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$. Define

$$\begin{aligned}
A &\coloneqq S^1 \cap (-\infty, 1/2) \times \mathbb{R} \subset S^1 \subset \mathbb{R}^2 \\
B &\coloneqq S^1 \cap (-1/2, +\infty) \times \mathbb{R} \subset S^1 \subset \mathbb{R}^2 \\
C &\coloneqq A \cap B \cap (\mathbb{R} \times (0, +\infty)) \\
D &\coloneqq A \cap B \cap (\mathbb{R} \times (-\infty, 0)) \\
U &\coloneqq A \times S^1 \\
V &\coloneqq B \times S^1.
\end{aligned}$$

Then, $T^2 = S^1 \times S^1 = (A \cup B) \times S^1 = A \times S^1 \cup B \times S^1 = U \cup V$. U and V are open in T^2, because A and B are open in $\mathbb{R}$ and by definition of the product topology.

We now prove an auxiliary result: for any $\theta_0, \theta_1 \in \mathbb{R}$ such that $0 < \theta_0 < \theta_1 < 2\pi$, if $L \coloneqq \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid \theta \in (\theta_0, \theta_1)\}$, then L is homotopy equivalent to a point. To show this, define $\theta_* = \theta_0/2 + \theta_1/2$. Define $p_* = (\cos\theta_*, \sin\theta_*)$ and

$$\begin{aligned}
f \colon \{p_*\} &\longrightarrow L & \qquad g \colon L &\longrightarrow \{p_*\} \\
p_* &\longmapsto p_*, & p &\longmapsto p_*.
\end{aligned}$$

Then, $g \circ f = \mathrm{id}_{\{p_*\}}$. It remains to show that $f \circ g$ is homotopic to id_L. For this, define

$$\begin{aligned}
H \colon \qquad [0,1] \times L &\longrightarrow L \\
(t, (\cos\theta, \sin\theta)) &\longmapsto (\cos((1-t)\theta_* + t\theta), \sin((1-t)\theta_* + t\theta)).
\end{aligned}$$

Then, H is a homotopy from id_L to $f \circ g$. So, L is homotopy equivalent to a point.

A, C, D satisfy the same conditions as L, so A, C and D are all homotopy equivalent to a point. B is homeomorphic to A (the map $(x, y) \longmapsto (-x, y)$ is a homeomorphism), so B is homotopy equivalent to a point as well.

We show that U and V are homotopy equivalent to S^1.

$$\begin{aligned}
U &= A \times S^1 && \text{[definition of } U\text{]} \\
&\simeq \{p\} \times S^1 && [A \simeq \{p\}\text{, products of htpy. equivalent spaces are htpy. equivalent]} \\
&\cong S^1,
\end{aligned}$$

and analogously $V \simeq S^1$.

We show that $U \cap V$ is homotopy equivalent to $S^1 \sqcup S^1$:

$$\begin{aligned}
U \cap V &= (A \times S^1) \cap (B \times S^1) && \text{[definition of } U, V\text{]} \\
&= (A \cap B) \times S^1 \\
&= (C \times S^1) \cup (D \times S^1) && \text{[definition of } C, D\text{]} \\
&\simeq (\{p\} \times S^1) \sqcup (\{p\} \times S^1) && \text{[products/disjoint unions of htpy. equivalent} \\
& && \text{spaces are htpy. equivalent]} \\
&\cong S^1 \sqcup S^1.
\end{aligned}$$

□

Exercise 13.2. Write the Mayer-Vietoris sequence for the decomposition of the torus in exercise 13.1. Use that the Betti numbers are invariant under homotopy equivalence.

Solution.

$$\begin{aligned}
&0 \longleftarrow H_0(X) \longleftarrow H_0(U) \oplus H_0(V) \longleftarrow H_0(U \cap V) \longleftarrow H_1(X) \longleftarrow \cdots \\
\Longrightarrow\ &[X = T^2 \text{ in our case}] \\
&0 \longleftarrow H_0(T^2) \longleftarrow H_0(U) \oplus H_0(V) \longleftarrow H_0(U \cap V) \longleftarrow H_1(T^2) \longleftarrow \cdots \\
\Longrightarrow\ &[U \simeq S^1, V \simeq S^1, U \cap V \simeq S^1 \sqcup S^1 \text{ and Homology is homotopy invariant}] \\
&0 \longleftarrow H_0(T^2) \longleftarrow H_0(S^1) \oplus H_0(S^1) \longleftarrow H_0(S^1 \sqcup S^1) \longleftarrow H_1(T^2) \longleftarrow \cdots \\
\Longrightarrow\ &[H_0(X \sqcup Y) = H_0(X) \oplus H_0(Y)] \\
&0 \longleftarrow H_0(T^2) \longleftarrow H_0(S^1) \oplus H_0(S^1) \longleftarrow H_0(S^1) \oplus H_0(S^1) \longleftarrow H_1(T^2) \longleftarrow \cdots \\
\Longrightarrow\ &[\text{By lemma 1.1 in lecture notes No. 9, } H_0(T^2) = \mathbb{R}. \\
&\ \text{By corollary 1.8 in lecture notes No. 9, } H_0(S^1) = H_1(S^1) = \mathbb{R}] \\
&0 \longleftarrow \mathbb{R} \longleftarrow \mathbb{R} \oplus \mathbb{R} \longleftarrow \mathbb{R} \oplus \mathbb{R} \longleftarrow H_1(T^2) \longleftarrow \cdots.
\end{aligned}$$

□

Exercise 13.3. Show that $b_1(T^2) \neq 0$.

Solution. By exercise 13.2, we have an exact sequence

$$0 \xleftarrow{f_0} \mathbb{R} \xleftarrow{g_0} \mathbb{R}^2 \xleftarrow{h_0} \mathbb{R}^2 \xleftarrow{f_1} H_1(T^2) \longleftarrow \cdots .$$

$$\begin{aligned}
\dim \operatorname{im} f_1 &= \dim \ker h_0 && [\text{exactness at } \mathbb{R}^2] \\
&= \dim \mathbb{R}^2 - \dim \operatorname{im} h_0 && [\text{rank-nullity theorem on } h_0] \\
&= \dim \mathbb{R}^2 - \dim \ker g_0 && [\text{exactness at } \mathbb{R}^2] \\
&= \dim \operatorname{im} g_0 && [\text{rank-nullity theorem on } g_0] \\
&= \dim \ker f_0 && [\text{exactness at } \mathbb{R}] \\
&= 1 && [f_0 \colon \mathbb{R} \longrightarrow \{0\}].
\end{aligned}$$

$$\begin{aligned}
b_1(T^2) &= \dim H_1(T^2) && [\text{definition of } b_1] \\
&= \dim \ker f_1 + \dim \operatorname{im} f_1 && [\text{rank-nullity on } f_1] \\
&= \dim \ker f_1 + 1 && [\text{computation above}] \\
&\neq 0 && [\dim \ker f_1 \geq 0].
\end{aligned}$$

□

Exercise 13.4. Show that T^2 and S^2 are not homotopy equivalent.

Solution. Assume by contradiction that they are.

$$\begin{aligned}
0 \neq b_1(T^2) && [\text{exercise 13.3}] \\
= b_1(S^2) && [\text{by assumption, } T^2 \simeq S^2, \text{ and by homotopy invariance of } b_1] \\
= 0 && [\text{by corollary 1.8 in lecture notes No. 9}].
\end{aligned}$$

Contradiction.

□

14 Exercise sheet No. 14

Exercise 14.1. Assume that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic. Show that $n = m$.

Solution. We use the symbols $\cong$ for "is homeomorphic to" and $\simeq$ for "has the same homotopy type as" and "is homotopic to".

We prove the result in the case where $n = 0$ or $m = 0$. If $n = 0$, then $\mathbb{R}^n = \mathbb{R}^0 = \{0\}$ has exactly one point. Since $\mathbb{R}^n \cong \mathbb{R}^m$, $\mathbb{R}^m$ has exactly one point as well. If $m \geq 1$, $\mathbb{R}^m$ would have more than one point, so $m = 0$. Analogously, we can show that $m = 0$ implies $n = 0$.

We prove the result in the case where $n \geq 1$ and $m \geq 1$. Step 1: we show that $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{0\}$. Let $\phi \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a homeomorphism. Define homeomorphisms $\phi|_{\mathbb{R}^n \setminus \{0\}} \colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{\phi(0)\}$ and $\psi \colon \mathbb{R}^m \setminus \{\phi(0)\} \longrightarrow \mathbb{R}^m \setminus \{0\}$ by $\psi(x) = x - \psi(0)$. Then, $\psi \circ \phi|_{\mathbb{R}^n \setminus \{0\}} \colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^m \setminus \{0\}$ is a homeomorphism.

Step 2: we show that $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$ and $S^{m-1} \simeq \mathbb{R}^m \setminus \{0\}$. For this, define

$$\begin{aligned} f \colon \mathbb{R}^n \setminus \{0\} &\longrightarrow S^{n-1} \\ x &\longmapsto \frac{x}{\|x\|}, \end{aligned} \qquad \begin{aligned} g \colon S^{n-1} \setminus \{0\} &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ x &\longmapsto x. \end{aligned}$$

Then, $f \circ g = \mathrm{id}_{S^{n-1}}$ and using the homotopy

$$\begin{aligned} H \colon [0,1] \times \mathbb{R}^n \setminus \{0\} &\longrightarrow \mathbb{R}^n \setminus \{0\} \\ (t, x) &\longmapsto (1-t)x + t\frac{x}{\|x\|} \end{aligned}$$

we conclude that $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$. So $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$. Analogously we show that $S^{m-1} \simeq \mathbb{R}^m \setminus \{0\}$.

Step 3: we show that $\forall k \in \mathbb{N}_0 \colon b_k(S^{n-1}) = b_k(S^{m-1})$.

$$\begin{aligned} &\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^m \setminus \{0\} && \text{[by Step 1]} \\ &\quad\Longrightarrow S^{n-1} \simeq S^{m-1} && \text{[by Step 2]} \\ &\quad\Longrightarrow \forall k \in \mathbb{N}_0 \colon b_k(S^{n-1}) = b_k(S^{m-1}) && \text{[Betti numbers are homotopy invariant].} \end{aligned}$$

Step 4: we show that if $n = 1$ or $m = 1$ then $n = m$. If $n = 1$, then

$$\begin{aligned} 2 &= b_0(S^0) && \text{[by corollary 1.8 in lecture notes No. 9]} \\ &= b_0(S^{n-1}) && [n = 1] \\ &= b_0(S^{m-1}) && \text{[by step 3]} \\ &\in \begin{cases} \{2\} & \text{if } m = 1 \\ \{0, 1\} & \text{if } m > 1 \end{cases} && \text{[by corollary 1.8 in lecture notes No. 9],} \end{aligned}$$

which implies that $m = 1$. Analogously, $m = 1$ implies that $n = 1$.

Step 5: we show that if $n > 1$ and $m > 1$ then $n = m$. Using step 3, $n > 1$ and $m > 1$, and corollary 1.8 in lecture notes No. 9 we conclude that $\forall k \in \mathbb{N}_0$:

$$\begin{cases} 1 & \text{if } k = 0, n-1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } k = 0, m-1 \\ 0 & \text{otherwise.} \end{cases}$$

This can be rewritten by saying that the following two sequences are equal:

$$(\underbrace{1}_{0}, 0, \dots, 0, \underbrace{1}_{n-1}, 0, \dots) = (\underbrace{1}_{0}, 0, \dots, 0, \underbrace{1}_{m-1}, 0, \dots).$$

This implies that $n = m$. $\square$