Riemannian Geometry - solutions to exercises

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Contents

0 Review of Lie groups and Lie algebras - 05-11-2020

0.1 Lie algebras

Definition 0.1 (Lie algebra)**.** A **Lie algebra** is a pair $(\mathfrak{g}, [\cdot,\cdot])$ where $\mathfrak{g}$ is a vector space over $\mathbb{R}$ and $[\cdot,\cdot]$ is a map $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies:

(a) (Skew-symmetric) $\forall x, y \in \mathfrak{g}\colon [x,y] = -[y,x]$;

(b) (Bilinear) $\forall a, b \in \mathbb{R}\colon \forall x,y,z \in \mathfrak{g}\colon [ax+by,z] = a[x,z]+b[y,z]$;

(c) (Jacobi identity) $\forall x,y,z \in \mathfrak{g}\colon [x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$.

The map $[\cdot,\cdot]$ is called the **Lie bracket** of $\mathfrak{g}$.

Example 0.2 (Lie algebras)**.**

(a) If M is a smooth manifold, then the set of vector fields on M, $\mathfrak{X}(M)$, together with the Lie bracket of M is a Lie algebra.

(b) Let V be a vector space over $\mathbb{R}$. Then, $\mathfrak{gl}(V) \coloneqq \mathrm{End}(V)$, together with the Lie bracket defined by

$$\forall T, S \in \mathrm{End}(V)\colon [T,S] = TS - ST,$$

is a Lie algebra.

0.2 Lie groups

0.2.1 Definitions and examples

Definition 0.3 (Lie group)**.** A **Lie group** is a group G which is at the same time a smooth manifold, such that the maps

$$\begin{aligned} G\times G &\longrightarrow G & G &\longrightarrow G \\ (g,h) &\longmapsto gh, & g &\longmapsto g^{-1}, \end{aligned}$$

are smooth.

Example 0.4 (Lie groups)**.** Consider the following groups of matrices:

$$\begin{aligned} \mathrm{GL}(n,\mathbb{R}) &= \left\{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\right\}, \\ \mathrm{SL}(n,\mathbb{R}) &= \left\{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\right\}, \\ \mathrm{O}(n) &= \left\{A \in \mathbb{R}^{n\times n} \mid A^\top A = \mathbb{1}_{n\times n}\right\}, \\ \mathrm{Sp}(2n,\mathbb{R}) &= \left\{A \in \mathbb{R}^{2n\times 2n} \mid A^\top J_0 A = J_0\right\}, \end{aligned}$$

where

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

We will see in a future exercise sheet that each of them is a submanifold of $\mathbb{R}^{n\times n}$ (and $\mathbb{R}^{2n\times 2n}$ in the case of $\mathrm{Sp}(2n,\mathbb{R})$). In each of these cases, the operations of multiplication and taking inverses are smooth. So, all the matrix subgroups presented above are matrix subgroups.

0.2.2 Lie algebra of a Lie group

Definition 0.5 (left invariant vector field)**.** Let G be a Lie group. For each $g \in G$, define the **left translation map** by

$$\begin{aligned} L_g \colon G &\longrightarrow G \\ h &\longmapsto gh. \end{aligned}$$

This map is a diffeomorphism with inverse $L_{g^{-1}}$. A vector field $X \in \mathfrak{X}(G)$ is **left invariant** if

$$\forall g \in G \colon (L_g)_* X = X.$$

Define the **set of left invariant vector fields** of G by

$$\mathfrak{X}_L(G) \coloneqq \{X \in \mathfrak{X}(G) \mid X \text{ is left invariant}\}.$$

Remark 0.6 (right invariant vector fields)**.** Analogously it's possible to define a right translation map and right invariant vector fields.

Proposition 0.7 (properties of left invariant vector fields)**.** *Let G be a Lie group. Then,*

(a) $\mathfrak{X}_L(G)$ *is a linear subspace of* $\mathfrak{X}(G)$*;*

(b) *The maps*

$$\begin{aligned} \phi_G \colon \mathfrak{X}_L(G) &\longrightarrow T_e G \\ X &\longmapsto X|_e \end{aligned}$$

$$\begin{aligned} \psi_G \colon \quad T_e G &\longrightarrow \mathfrak{X}_L(G) \\ V &\longmapsto X^V, \ \text{where } X^V_g = \mathrm{D}L_g(e)V \end{aligned}$$

are linear and inverses of one another, hence isomorphisms of vector spaces;

(c) $\dim \mathfrak{X}_L(G) = \dim G$*;*

(d) $\mathfrak{X}_L(G)$ *is a Lie subalgebra of* $\mathfrak{X}(G)$.

Proof. Exercise. ☐

Definition 0.8 (Lie algebra of a Lie group)**.** Let G be a Lie group. Define $\mathfrak{g} = T_e G$ as a vector space. Define on $\mathfrak{g}$ a Lie bracket $[\cdot,\cdot]$ by carrying over the Lie bracket of $\mathfrak{X}_L(G)$ with the isomorphism $\mathfrak{g} = T_e G \cong \mathfrak{X}_L(G)$, or in other words, such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "\psi_G \times \psi_G"] \arrow[d, "{[\cdot,\cdot]}"'] & \mathfrak{X}_L(G) \times \mathfrak{X}_L(G) \arrow[d, "{[\cdot,\cdot]}"] \\ \mathfrak{g} & \mathfrak{X}_L(G) \arrow[l, "\phi_G"] \end{tikzcd} .$$

Then $\mathfrak{g}$ is a Lie algebra, called the **Lie algebra** of G.

Example 0.9 (Lie algebra of $\mathrm{GL}(n,\mathbb{R})$)**.** The Lie algebra of the Lie group $\mathrm{GL}(n,\mathbb{R})$ is $\mathfrak{gl}(\mathbb{R}^n)$.

0.2.3 Exponential map

Definition 0.10 (exponential map)**.** Let G be a Lie group. The **exponential map** of G is a map $\exp\colon \mathfrak{g} \longrightarrow G$ given by

$$\exp(V) = \phi^1_{X^V}(e),$$

where X^V is the left invariant vector field which is equal to V at the identity and $\phi^1_{X^V}$ is the time 1 flow of this vector field.

Proposition 0.11 (properties of exponential map)**.** *Let G be a Lie group. Then, the exponential map of G has the following properties:*

(a) $\forall t, s \in \mathbb{R}\colon \forall V \in \mathfrak{g}\colon\ \exp((t+s)V) = \exp(tV)\exp(sV)$;

(b) $\forall t \in \mathbb{R}\colon \forall V \in \mathfrak{g}\colon\ \exp(-tV) = (\exp(tV))^{-1}$;

(c) $\exp$ *is smooth and* $\mathrm{D}\exp(0) = \mathrm{id}_{\mathfrak{g}}$.

0.3 Lie group actions

0.3.1 Definition and examples

Definition 0.12 (Lie group action)**.** Let G be a Lie group and M be a manifold. A **left action** of G on M is a smooth map $G \times M \longrightarrow M$, $(g, p) \longmapsto gp$, such that

- $ep = p$, for all $p \in M$;
- $g(hp) = (gh)p$, for all $g, h \in G$ and $p \in M$.

By the properties of the action, since $g(hp) = (gh)p$ there is no ambiguity in the expression ghp, so we typically omit the parenthesis.

Remark 0.13 (right actions)**.** Analogously, it's possible to define a right action $M \times G \longrightarrow M$, which we denote with multiplication on the right, i.e. $(p, g) \longmapsto pg$ and which satisfies $p(gh) = (pg)h$.

Definition 0.14 (orbit and isotropy)**.** Let G be a Lie group, M be a manifold and $G \times M \longrightarrow M$ be a left action of G on M. Define

- for each $x \in M$, the **orbit of** x:

$$\mathcal{O}_x = \{y \in M \mid \exists g \in G\colon gx = y\};$$

- for each $x \in M$, the **isotropy subgroup of** x:

$$G_x = \{g \in G \mid gx = x\}.$$

Example 0.15 (Lie group actions)**.**

(a) If G is a Lie group, then there is an action by **left translation**

$$\begin{aligned} L\colon G \times G &\longrightarrow G \\ (g, h) &\longmapsto L_g(h) = gh. \end{aligned}$$

(b) If G is a Lie group, then there is an action by **conjugation**

$$\begin{aligned} C\colon G\times G &\longrightarrow G \\ (g,h) &\longmapsto C_g(h)=ghg^{-1}. \end{aligned}$$

This action satisfies

(b.1) $C_{gh}=C_gC_h$ (just restating the fact that it is an action);

(b.2) $C_g(ab)=C_g(a)C_g(b)$ ($C_g\colon G\longrightarrow G$ is not just a diffeomorphism but a Lie group homeomorphism);

(b.3) $(C_g)^{-1}=C_{g^{-1}}$ (C_g is a Lie group isomorphism).

0.3.2 Quotient of a manifold by a group action

Definition 0.16 (M/G as a set)**.** Let $G\times M\longrightarrow M$ be a Lie group action. Define an equivalence relation $\sim$ by

$$p\sim q \Longleftrightarrow \exists g\in G\colon q=gp.$$

Then, the equivalence class of p, $[p]$ is equal to the orbit of p, G_p. Define the **quotient** of M by the Lie group (as a set) by

$$\begin{aligned} M/G &= M/\sim \\ &= \{[x]\mid x\in M\} \\ &= \{\mathcal{O}_x\mid x\in M\}. \end{aligned}$$

Definition 0.17 (free Lie group action)**.** A Lie group action $G\times M\longrightarrow M$ is **free** if for all $x\in M$ we have that $G_x=\{e\}$.

Definition 0.18 (proper Lie group action)**.** A Lie group action $G\times M\longrightarrow M$ is **proper** if the map

$$\begin{aligned} G\times M &\longrightarrow M\times M \\ (g,p) &\longmapsto (gp,p) \end{aligned}$$

is proper, i.e. the preimage of a compact set is compact.

Theorem 0.19 (M/G is a manifold)**.** *Let $G\times M\longrightarrow M$ be a Lie group action. If the action is free and proper, then M/G has the structure of a smooth manifold such that the quotient map $\pi\colon M\longrightarrow M/G$ is smooth and a submersion.*

0.3.3 Coverings

The following facts are stated in [GN14].

Definition 0.20 (covering)**.** Let $\pi\colon M\longrightarrow B$ be a smooth map. π is a **covering map** if M is connected and for every $p\in B$, there exists U a neighbourhood of p in B and a family of open sets U_α (for α in an index set I) such that $\pi^{-1}(U)=\bigcup_{\alpha\in I}U_\alpha$ and for every α we have that $\pi|_{U_\alpha}\colon U_\alpha\longrightarrow U$ is a diffeomorphism.

Definition 0.21 (deck transformations)**.** Let $\pi\colon M \longrightarrow B$ be a covering. A **deck transformation** of π is a diffeomorphism $h\colon M \longrightarrow M$ such that $\pi \circ h = \pi$. The group of deck transformations is the Lie group

$$G = \{h\colon M \longrightarrow M \mid h \text{ is a diffeomorphism}, \pi \circ h = \pi\}$$

which is equipped with the discrete topology (this uniquely determines the manifold structure of the Lie group).

Definition 0.22 (universal covering)**.** A covering $\pi\colon M \longrightarrow B$ is a **universal covering** if M is simply connected.

Theorem 0.23 (deck transformations of a universal covering)**.** *Let $\pi\colon M \longrightarrow B$ be a universal covering with group of deck of transformations G. Then,*

(a) *G acts on M via $h \cdot x = h(x)$, and this action is free and proper.*

(b) *There exists a unique map such that the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\pi_G"'] & B \\ M/G \arrow[ur, dashed, "\exists!\phi"'] & \end{tikzcd},$$

and this map ϕ is a diffeomorphism.

(c) *G is isomorphic as a group to $\pi_1(B)$.*

Theorem 0.24 (Lie)**.** *Let $\mathfrak{g}$ be a Lie algebra. Then, there exists a unique Lie group $\tilde{G}$ which is simply connected and such that the Lie algebra of $\tilde{G}$ is $\mathfrak{g}$.*

Theorem 0.25 (universal covering of a Lie group)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\tilde{G}$ be the unique simply connected Lie group with Lie algebra $\mathfrak{g}$ (coming from theorem 0.24). Then, there exists a unique $\pi\colon \tilde{G} \longrightarrow G$ which is a covering map and a group homomorphism. In addition, π satisfies $G \cong \tilde{G}/\ker\pi$ and $\ker\pi$ is isomorphic to the group of deck transformations of the covering $\pi\colon \tilde{G} \longrightarrow G$.*

1 Exercise sheet No. 1 - 12-11-2020

Exercise 1.1 (connected and path connected)**.** Prove that if M is a connected topological manifold then M is path-connected.

Solution. Let n be the dimension of M. It suffices to assume that $p \in M$, $S = \{q \in M \mid \text{there exists a path } \gamma \text{ from } p \text{ to } q\}$, and to prove that $S = M$. For this, since M is connected it suffices to show that S is nonempty, open and closed. S is nonempty, because $p \in S$ (the constant path at p is a path from p to p).

We show that S is open. For this, it suffices to assume that $q \in S$ and to prove that there exists $U \subset M$ open such that $q \in U \subset S$. We claim that there exist $U \subset M$ an open neighbourhood of q, $O \subset \mathbb{R}^n$ open, and a homeomorphism $\sigma \colon U \longrightarrow O$ such that U and O are path connected. This is because since M is a topological manifold, it is locally Euclidean, and therefore such U, O, σ exist with O possibly not path connected. By restricting U and O, we can assume that O is a ball. So, both U and O are path connected. We claim that $U \subset S$. To show this, it suffices to assume that $x \in U$ and to prove that $x \in S$. Since $q \in S$, there exists a path γ_1 from p to q. Since U is path connected and $x, q \in U$, there exists a part γ_2 from q to x. So, the concatenation of γ_1 and γ_2 is a path from p to x. So $x \in S$. This concludes the proof that S is open.

We show that S is closed. Let $R = M \setminus S$. It suffices to show that R is open. For this, it suffices to assume that $q \in R$ and to prove that there exists $U \subset M$ open such that $q \in U \subset R$. Proceeding as above, we can conclude that there exist $U \subset M$ an open neighbourhood of q, $O \subset \mathbb{R}^n$ open, and a homeomorphism $\sigma \colon U \longrightarrow O$ such that U and O are path connected. We claim that $U \subset R$. To show this, it suffices to assume that $x \in U$ and to prove that $x \in R$. Assume by contradiction that $x \notin R$, in other words $x \in S$. Then, there exists a path γ_1 from p to x. Since U is path connected and $x, q \in U$, there exists a part γ_2 from x to q. So, the concatenation of γ_1 and γ_2 is a path from p to q. So $q \in S$, but by assumption $q \in R = M \setminus S$. Contradiction. This concludes the proof that S is closed, and the proof that M is path-connected. □

Exercise 1.2 (immersions and embeddings)**.** Show that

(a) if X is a compact topological space, Y is a Hausdorff topological space, and $f \colon X \longrightarrow Y$ is continuous and bijective, then f is a homeomorphism;

(b) if M is a compact smooth manifold, N is a smooth manifold, and $f \colon M \longrightarrow N$ is an injective immersion, then f is an embedding.

Solution. (a): It suffices to show that f^{-1} is continuous. For this, it suffices to assume that $U \subset X$ is open and to prove that $f(U) \subset Y$ is open.

$$\begin{aligned}
&U \subset X \text{ is open} \\
&\quad\implies [\text{by definition of closed set}] \\
&\quad\quad X \setminus U \subset X \text{ is closed} \\
&\quad\implies [\text{a closed subset of a compact set is compact}] \\
&\quad\quad X \setminus U \subset X \text{ is compact} \\
&\quad\implies [\text{image of compact set under continuous map is compact}] \\
&\quad\quad f(X \setminus U) = Y \setminus f(U) \subset Y \text{ is compact}
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \text{[a compact subset of a Hausdorff space is closed]} \\
&\ Y \setminus f(U) \subset Y \text{ is closed} \\
&\Longrightarrow \text{[by definition of closed set]} \\
&\ f(U) \text{ is open.}
\end{aligned}$$

(b): By definition of embedding, it suffices to show that $f\colon M \longrightarrow N$ is a homeomorphism in its image, in other words that $f\colon M \longrightarrow f(M)$ is a homeomorphism. This follows immediately from (a). ☐

Exercise 1.3 (matrix Lie groups)**.** Prove the following.

(a) Show that the general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \left\{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\right\}$$

is a sumbanifold of $\mathbb{R}^{n\times n}$. What is the dimension?

(b) Show that the special linear group

$$\mathrm{SL}(n, \mathbb{R}) = \left\{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\right\}$$

is a sumbanifold of $\mathbb{R}^{n\times n}$. What is the dimension?

(c) Show that the orthogonal group

$$\mathrm{O}(n) = \left\{A \in \mathbb{R}^{n\times n} \mid A^\top A = \mathbb{1}_{n\times n}\right\}$$

is a sumbanifold of $\mathbb{R}^{n\times n}$. What is the dimension?

(d) Show that the symplectic group

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{A \in \mathbb{R}^{2n\times 2n} \mid A^\top J_0 A = J_0\right\},$$

where

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}$$

is a sumbanifold of $\mathbb{R}^{2n\times 2n}$. What is the dimension?

Solution. (a): $\det\colon \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is continuous and $\mathbb{R} \setminus \{0\}$ is open. So, $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$. An open subset of a manifold is a manifold of the same dimension.

(b): Consider the determinant map $\det\colon \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$. Then, det is a smooth map (because the formula that defines the determinant of a matrix shows that it depends smoothly on the entries of the matrix.) and $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$. We claim that 1 is a regular value of det. To show this, it suffices to assume that $A \in \mathbb{R}^{n\times n}$, that $\det(A) = 1$ and to prove that $\mathrm{D}\det(A) \neq 0$. For any $V \in \mathbb{R}^{n\times n}$, by Jacobi's formula (a formula for the derivative of the determinant)

$$\begin{aligned}
\mathrm{D}\det(A)V &= \det(A)\operatorname{tr}(A^{-1}V) && \text{[by Jacobi's formula]} \\
&= \operatorname{tr}(A^{-1}V) && [\det A = 1].
\end{aligned}$$

So, for $V = A$, $\mathrm{D}\det(A)A = \mathrm{tr}(A^{-1}A) = n \neq 0$. So $\mathrm{D}\det(A) \neq 0$. We conclude that 1 is a regular value of det. By the theorem about preimages of regular values, $\mathrm{SL}(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{n\times n}$ of dimension $n \times n - 1$.

(c): Define $\mathrm{Symm}(n) \subset \mathbb{R}^{n\times n}$ to be the subset of those matrices which are symmetric. Define $f \colon \mathbb{R}^{n\times n} \longrightarrow \mathrm{Symm}(n)$ by $f(A) = A^T A$. Then, f is well defined (because if A is any matrix, $A^T A$ is symmetric), smooth and $\mathrm{O}(n, \mathbb{R}) = f^{-1}(\mathbb{1}_{n\times n})$. We claim that $\mathbb{1}_{n\times n}$ is a regular value of f. To show this, it suffices to assume that $A \in \mathbb{R}^{n\times n}$, that $f(A) = \mathbb{1}_{n\times n}$, and to prove that $\mathrm{D}f(A) \colon \mathbb{R}^{n\times n} \longrightarrow \mathrm{Symm}(n)$ is surjective. For this, it suffices to assume that $S \in \mathrm{Symm}(n)$ and to prove that there exists $V \in \mathbb{R}^{n\times n}$ such that $\mathrm{D}f(A)V = S$. We start by computing $\mathrm{D}f(A)$:

$$\begin{aligned}
\mathrm{D}f(A)V &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(A + tV) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (A^T + tV^T)(A + tV) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (A^T A + tA^T V + tV^T A + t^2 V^T V) \\
&= A^T V + V^T A.
\end{aligned}$$

Define $V = \frac{1}{2} AS$. We show that V is as desired:

$$\begin{aligned}
\mathrm{D}f(A)V &= \frac{1}{2}(A^T AS + (AS)^T A) && [\text{definition of } V] \\
&= \frac{1}{2}(A^T AS + S^T A^T A) && [\text{transpose of product of matrices}] \\
&= \frac{1}{2}(S + S) && [S \text{ is symmetric and } A \text{ is orthogonal}] \\
&= S.
\end{aligned}$$

We conclude that $\mathbb{1}_{n\times n}$ is a regular value of f. By the theorem about preimages of regular values, $\mathrm{O}(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{n\times n}$ of dimension $n^2 - \dim \mathrm{Symm}(n) = n^2 - n(n + 1)/2 = n(n - 1)/2$.

(d): Define $\mathrm{Asymm}(2n) \subset \mathbb{R}^{2n\times 2n}$ to be the subset of those matrices which are anti-symmetric. Define $f \colon \mathbb{R}^{2n\times 2n} \longrightarrow \mathrm{Asymm}(2n)$ by $f(A) = A^T J_0 A$. Then, f is well defined (because if A is any matrix, $A^T J_0 A$ is anti-symmetric), smooth and $\mathrm{Sp}(2n, \mathbb{R}) = f^{-1}(J_0)$. We claim that J_0 is a regular value of f. To show this, it suffices to assume that $A \in \mathbb{R}^{2n\times 2n}$, that $f(A) = J_0$, and to prove that $\mathrm{D}f(A) \colon \mathbb{R}^{2n\times 2n} \longrightarrow \mathrm{Asymm}(2n)$ is surjective. For this, it suffices to assume that $S \in \mathrm{Asymm}(2n)$ and to prove that there exists $V \in \mathbb{R}^{2n\times 2n}$ such that $\mathrm{D}f(A)V = S$. We start by computing $\mathrm{D}f(A)$:

$$\begin{aligned}
\mathrm{D}f(A)V &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(A + tV) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (A^T + tV^T) J_0 (A + tV) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (A^T J_0 A + tA^T J_0 V + tV^T J_0 A + t^2 V^T J_0 V) \\
&= A^T J_0 V + V^T J_0 A.
\end{aligned}$$

Define $V = -\frac{1}{2} A J_0 S$. We show that V is as desired:

$$\mathrm{D}f(A)V$$

$$
\begin{aligned}
&= -\frac{1}{2}(A^T J_0(AJ_0S) + (AJ_0S)^T J_0A) && [\text{definition of } V] \\
&= -\frac{1}{2}(A^T J_0 A J_0 S + S^T J_0^T A^T J_0 A) && \\
&= -\frac{1}{2}(J_0J_0S + SJ_0J_0) && [f(A) = J_0, S \in \operatorname{Asymm}(2n)] \\
&= S && [J_0^2 = 0].
\end{aligned}
$$

Therefore J_0 is a regular value of f. By the theorem about preimages of regular values, $\operatorname{Sp}(2n)$ is a submanifold of $\mathbb{R}^{2n\times 2n}$ of dimension $(2n)^2 - \dim \operatorname{Asymm}(2n) = (2n)^2 - 2n(2n-1)/2 = 2n^2 + n$. □

Exercise 1.4 (Lie bracket in coordinates)**.** Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$ be vector fields in M. Let $(x^1, \ldots, x^n)$ be a coordinate chart on M. Show that with respect to these coordinates, $[X, Y] \in \mathfrak{X}(M)$ is given by

$$
[X, Y] = \sum_{i=1}^n \left(\sum_{j=1}^n \left(a^j(x) \frac{\partial b^i}{\partial x^j}(x) - b^j(x) \frac{\partial a^i}{\partial x^j}(x) \right) \right) \frac{\partial}{\partial x^i}.
$$

Solution. For any f a real valued function on M,

$$
\begin{aligned}
[X, Y]f &= \left[\sum_{i=1}^n a^i(x) \frac{\partial}{\partial x^i}, \sum_{j=1}^n b^j(x) \frac{\partial}{\partial x^j} \right] f \\
&= \sum_{i=1}^n \sum_{j=1}^n \left[a^i(x) \frac{\partial}{\partial x^i}, b^j(x) \frac{\partial}{\partial x^j} \right] f \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(a^i(x) \frac{\partial}{\partial x^i} \left(b^j(x) \frac{\partial f}{\partial x^j} \right) - b^j(x) \frac{\partial}{\partial x^j} \left(a^i(x) \frac{\partial f}{\partial x^i} \right) \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(a^i(x) \frac{\partial b^j(x)}{\partial x^i} \frac{\partial f}{\partial x^j} + a^i(x) b^j(x) \frac{\partial^2 f}{\partial x^i \partial x^j} \right. \\
&\qquad\qquad \left. - b^j(x) \frac{\partial a^i(x)}{\partial x^j} \frac{\partial f}{\partial x^i} - a^i(x) b^j(x) \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(a^i(x) \frac{\partial b^j(x)}{\partial x^i} \frac{\partial f}{\partial x^j} - b^j(x) \frac{\partial a^i(x)}{\partial x^j} \frac{\partial f}{\partial x^i} \right) \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n \left(a^j(x) \frac{\partial b^i}{\partial x^j}(x) - b^j(x) \frac{\partial a^i}{\partial x^j}(x) \right) \right) \frac{\partial}{\partial x^i} f.
\end{aligned}
$$
□

Exercise 1.5 (coordinate change of vector)**.** Let M be a smooth manifold, $p \in M$ and $v \in T_pM$. Let $\phi = (x^1, \ldots, x^n)$, $\psi = (y^1, \ldots, y^n)$ be coordinate charts around p on M, with respect to which v is written as

$$
v = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\bigg|_p, \quad v = \sum_{i=1}^n b^i \frac{\partial}{\partial y^i}\bigg|_p.
$$

Show that

$$
a^j = \sum_{i=1}^n b^i \frac{\partial x^j}{\partial y^i}.
$$

Solution. For $j = 1, \ldots, n$, define curves γ_x^j and γ_y^j by

$$\begin{aligned} \phi \circ \gamma_x^j(t) &= (x^1(p), \ldots, x^{j-1}(p), x^j(p) + t, x^{j+1}(p), \ldots, x^n(p)), \\ \phi \circ \gamma_y^j(t) &= (y^1(p), \ldots, y^{j-1}(p), y^j(p) + t, y^{j+1}(p), \ldots, y^n(p)). \end{aligned}$$

Then, by definition of the vectors $\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j} \in T_pM$, $\frac{\partial}{\partial x^j}\Big|_p = \dot{\gamma}_x^j(0)$ and $\frac{\partial}{\partial y^j}\Big|_p = \dot{\gamma}_y^j(0)$. We show that $\frac{\partial}{\partial y^i}\Big|_p = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}\Big|_p$. For any function f,

$$\begin{aligned} &\sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}(f) \\ &\quad = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \gamma_x^j)(t) && [\text{definition of } \tfrac{\partial}{\partial x^j}] \\ &\quad = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (f \circ \phi^{-1} \circ \phi \circ \gamma_x^j)(t) \\ &\quad = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}(f \circ \phi^{-1}) && [\text{definition of partial derivative}] \\ &\quad = \frac{\partial}{\partial y^i}(f \circ \psi^{-1}) && [\text{chain rule for maps between Eucl. spaces}] \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \psi^{-1} \circ \psi \circ \gamma_y^i && [\text{definition of partial derivative}] \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \gamma_y^i \\ &\quad = \frac{\partial}{\partial y^i}(f) && [\text{definition of } \tfrac{\partial}{\partial y^i}]. \end{aligned}$$

We now complete the proof:

$$\begin{aligned} \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\Big|_p &= v && [\text{by hypothesis}] \\ &= \sum_{i=1}^n b^i \frac{\partial}{\partial y^i}\Big|_p && [\text{by hypothesis}] \\ &= \sum_{i=1}^n b^i \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}\Big|_p && [\text{by the previous computation}] \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n b^i \frac{\partial x^j}{\partial y^i} \right) \frac{\partial}{\partial x^j}\Big|_p. \end{aligned}$$

We conclude that $a^j = \sum_{i=1}^n b^i \frac{\partial x^j}{\partial y^i}$ because both sides of this equality are the components of v in the basis $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$. □

Exercise 1.6 (related vector fields)**.** Let M, N be smooth manifolds and $\phi\colon M \longrightarrow N$ be a smooth map.

(a) Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be vector fields. Show that the following are equivalent:

(a.1) X is ϕ-related to Y, i.e. for all $p \in M$ we have that $\mathrm{D}\phi(p)X_p = Y_{\phi(p)}$;

(a.2) The following diagram commutes:

$$\begin{tikzcd} C^\infty(N,\mathbb{R}) \arrow[r, "\phi^*"] \arrow[d, "Y"'] & C^\infty(M,\mathbb{R}) \arrow[d, "X"] \\ C^\infty(N,\mathbb{R}) \arrow[r, "\phi^*"'] & C^\infty(M,\mathbb{R}) \end{tikzcd},$$

where $\phi^* f = f \circ \phi$, $X\colon C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ is the map $f \longmapsto X(f)$ and analogously for Y.

(b) For $i = 0,1$ let $X_i \in \mathfrak{X}(M)$ and $Y_i \in \mathfrak{X}(N)$ be vector fields. Show that if X_i is ϕ-related to Y_i for $i = 0,1$, then $[X_0, X_1]$ is ϕ-related to $[Y_0, Y_1]$.

Solution. (a):

$$\begin{aligned}
&X \circ \phi^* = \phi^* \circ Y \\
&\iff \forall f \in C^\infty(N,\mathbb{R})\colon X \circ \phi^*(f) = \phi^* \circ Y(f) \\
&\iff [\text{definition of the maps } \phi,\ X \text{ and } Y] \\
&\qquad \forall f \in C^\infty(N,\mathbb{R})\colon X(f\circ\phi) = Y(f)\circ\phi \\
&\iff [\text{two functions are equal if and only if they are equal at all points}] \\
&\qquad \forall f \in C^\infty(N,\mathbb{R})\colon \forall p \in M\colon X_p(f\circ\phi) = Y_{\phi(p)}(f) \\
&\iff [\text{for any function } g \text{ and vector field } Z, \text{ we have } Z_p(g) = \mathrm{D}g(p)Z_p] \\
&\qquad \forall f \in C^\infty(N,\mathbb{R})\colon \forall p \in M\colon \mathrm{D}(f\circ\phi)(p)X_p = \mathrm{D}f(\phi(p))Y_{\phi(p)} \\
&\iff [\text{Chain rule}] \\
&\qquad \forall f \in C^\infty(N,\mathbb{R})\colon \forall p \in M\colon \mathrm{D}f(\phi(p))\mathrm{D}\phi(p)X_p = \mathrm{D}f(\phi(p))Y_{\phi(p)} \\
&\iff [(\Longleftarrow)\text{: trivial. } (\Longrightarrow)\text{: because } f \text{ is arbitrary}] \\
&\qquad \forall p \in M\colon \mathrm{D}\phi(p)X_p = Y_{\phi(p)}.
\end{aligned}$$

(b): By (a), it suffices to show that $[X_0, X_1] \circ \phi^* = \phi^* \circ [Y_0, Y_1]$.

$$\begin{aligned}
[X_0,X_1]\circ\phi &= X_0\circ X_1\circ\phi^* - X_1\circ X_0\circ\phi^* && [\text{definition of Lie bracket}] \\
&= X_0\circ\phi^*\circ Y_1 - X_1\circ\phi^*\circ Y_0 && [X_i \text{ is } \phi\text{-related to } Y_i] \\
&= \phi^*\circ Y_0\circ Y_1 - \phi^*\circ Y_1\circ Y_0 && [X_i \text{ is } \phi\text{-related to } Y_i] \\
&= \phi^*\circ[Y_0,Y_1] && [\text{definition of Lie bracket}].
\end{aligned}$$

□

2 Exercise sheet No. 2 - 19-11-2020

Exercise 2.1 (wedge product and linear independence)**.** Let V be a finite dimensional vector space and $T_1, \dots, T_k \in V^*$. Show that $T_1, \dots, T_k$ are linearly independent if and only if $T_1 \wedge \cdots \wedge T_k \neq 0$.

Solution. ($\Longrightarrow$): Since $T_1, \dots, T_k$ are linearly independent, we can extend them to a basis $T_1, \dots, T_n$ of V^*. Let $v_1, \dots, v_n$ be the dual basis of V. Then,

$$\begin{aligned} T_1 \wedge \cdots \wedge T_n(v_1, \dots, v_n) &= \det([T_i(v_j)]_{ij}) \\ &= \det([\delta_{ij}]_{ij}) \qquad \text{[by definition of dual basis]} \\ &= 1 \\ &\neq 0. \end{aligned}$$

Therefore, $T_1 \wedge \cdots \wedge T_n \neq 0$ and $T_1 \wedge \cdots \wedge T_k \neq 0$.

($\Longleftarrow$): Assume by contradiction that $T_1, \dots, T_k$ are linearly dependent. Then, there exist $a_1, \dots, a_k \in \mathbb{R}$ such that $\sum_{i=1}^k a_i T_i = 0$ and a $j \in \{1, \dots, k\}$ such that $a_j \neq 0$. Then, $T_1 \wedge \cdots \wedge T_k = 0$:

$$\begin{aligned} T_1 \wedge \cdots \wedge T_k &= T_1 \wedge \cdots \wedge T_{j-1} \wedge T_j \wedge T_{j+1} \wedge \cdots \wedge T_k \\ &= T_1 \wedge \cdots \wedge T_{j-1} \wedge \left(\frac{1}{a_j} \sum_{i=1, i \neq j}^k a_i T_i \right) \wedge T_{j+1} \wedge \cdots \wedge T_k \\ &= \sum_{i=1, i\neq j}^k \frac{a^i}{a_j} T_1 \wedge \cdots \wedge T_{j-1} \wedge T_i \wedge T_{j+1} \wedge \cdots \wedge T_k \\ &= 0. \end{aligned}$$

Since by assumption $T_1 \wedge \cdots \wedge T_k \neq 0$, we obtain a contradiction. □

Exercise 2.2 (coordinate change of a tensor)**.** Let T be a $(2,1)$-tensor field on a smooth manifold M. Let (U, x^i) and (V, y^j) be two charts on M such that $U \cap V \neq \varnothing$. Here x^i and y^j represent the coordinates on U and V, respectively. Denote by ${}^yT^c_{ab}$ the components of T with respect to (V, y^j) and ${}^xT^k_{ij}$ the components of T with respect to the chart (U, x^i). Show that on the overlap $U \cap V$ we have

$${}^yT^c_{ab} = \frac{\partial y^c}{\partial x^k} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} {}^xT^k_{ij}.$$

Solution. It suffices to show that at a point $p \in U \cap V$,

$${}^yT^c_{ab}|_p = \left.\frac{\partial y^c}{\partial x^k}\right|_p \left.\frac{\partial x^i}{\partial y^a}\right|_p \left.\frac{\partial x^j}{\partial y^b}\right|_p {}^xT^k_{ij}|_p.$$

Notice that

$$\left\{ \left.\frac{\partial}{\partial y^c}\right|_p \otimes \mathrm{d}y^a|_p \otimes \mathrm{d}y^b|_p \right\}_{a,b,c}, \quad \left\{ \left.\frac{\partial}{\partial x^k}\right|_p \otimes \mathrm{d}x^i|_p \otimes \mathrm{d}x^j|_p \right\}_{i,j,k}$$

are bases for the vector space of $(2,1)$-tensors on T_pM. With respect to these bases,

$$T|_p = {}^xT^k_{ij}|_p \left.\frac{\partial}{\partial x^k}\right|_p \otimes \mathrm{d}x^i|_p \otimes \mathrm{d}x^j|_p,$$

$$T|_p = {}^yT^c_{ab}|_p \frac{\partial}{\partial y^c}\bigg|_p \otimes \mathrm{d}y^a|_p \otimes \mathrm{d}y^b|_p,$$

by definition of component functions of a tensor. As a consequence of the definition of $\frac{\partial}{\partial x^i}$, $\mathrm{d}x^i$, $\frac{\partial}{\partial y^a}$, $\mathrm{d}y^a$, it's possible to show that

$$\begin{aligned}
\frac{\partial}{\partial x^k} &= \frac{\partial y^c}{\partial x^k}\frac{\partial}{\partial y^c}, \\
\mathrm{d}x^i &= \frac{\partial x^i}{\partial y^a}\mathrm{d}y^a, \\
\mathrm{d}x^j &= \frac{\partial x^j}{\partial y^b}\mathrm{d}y^b.
\end{aligned}$$

Then,

$$\begin{aligned}
&{}^yT^c_{ab}|_p \frac{\partial}{\partial y^c}\bigg|_p \otimes \mathrm{d}y^a|_p \otimes \mathrm{d}y^b|_p \\
&\quad = T|_p \\
&\quad = {}^xT^k_{ij}|_p \frac{\partial}{\partial x^k}\bigg|_p \otimes \mathrm{d}x^i|_p \otimes \mathrm{d}x^j|_p \\
&\quad = {}^xT^k_{ij}|_p \frac{\partial y^c}{\partial x^k}\bigg|_p \frac{\partial x^i}{\partial y^a}\bigg|_p \frac{\partial x^j}{\partial y^b}\bigg|_p \frac{\partial}{\partial y^c}\bigg|_p \otimes \mathrm{d}y^a|_p \otimes \mathrm{d}y^b|_p.
\end{aligned}$$

In the computation, both the last term and the first are elements of the vector space of $(2,1)$-tensors on T_pM, written in the basis $\left\{ \frac{\partial}{\partial y^c}\Big|_p \otimes \mathrm{d}y^a|_p \otimes \mathrm{d}y^b|_p \right\}_{a,b,c}$. Since the vectors are equal, we conclude that their components in this basis are equal as well:

$${}^yT^c_{ab}|_p = \frac{\partial y^c}{\partial x^k}\bigg|_p \frac{\partial x^i}{\partial y^a}\bigg|_p \frac{\partial x^j}{\partial y^b}\bigg|_p {}^xT^k_{ij}|_p. \qquad \square$$

Exercise 2.3 (Homotopy invariace of integral, taken from [GN14])**.** Let M, N be smooth manifolds (without boundary) with $\dim M = n$, $\omega \in \Omega^n(N)$ be a closed form on N, $f_0, f_1 \colon M \longrightarrow N$ be smooth maps, and $H \colon [0,1] \times M \longrightarrow N$ be a smooth homotopy from f_0 to f_1 (i.e., $H(0,p) = f_0(p)$ and $H(1,p) = f_1(p)$). Show that $\int_M f_1^*\omega = \int_M f_0^*\omega$.

Solution. Define maps $\iota_0, \iota_1 \colon M \longrightarrow [0,1] \times M$ given by $\iota_i(p) = (i,p)$ for $i = 0,1$. Define diffeomorphisms $\phi_i \colon M \longrightarrow \{i\} \times M$ given by $\phi(p) = (i,p)$, for $i = 0,1$. Define also $\iota \colon \{0,1\} \times M \longrightarrow [0,1] \times M$ the inclusion. Notice that $\partial([0,1] \times M) = \{0,1\} \times M$ and that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\phi_i"] \arrow[rrd, bend right, "f_i"'] & \{i\} \times M \arrow[r, "\iota_i"] & {[0,1] \times M} \arrow[d, "H"] \\
 & & N
\end{tikzcd}.$$

$$\begin{aligned}
&\int_M f_1^*\omega - \int_M f_0^*\omega \\
&\quad = \int_M (H \circ \iota_1 \circ \phi_1)^*\omega - \int_M (H \circ \iota_0 \circ \phi_0)^*\omega \quad \text{[the diagram above commutes]}
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \phi_1^* \iota_1^* H^* \omega - \int_M \phi_0^* \iota_0^* H^* \omega && \text{[property of pullbacks]} \\
&= \int_{\{1\}\times M} \iota_1^* H^* \omega - \int_{\{0\}\times M} \iota_0^* H^* \omega && \text{[diffeo. invariance of integral]} \\
&= \int_{\{0,1\}\times M} \iota^* H^* \omega && \text{[additivity of integral]} \\
&= \int_{\partial([0,1]\times M)} \iota^* H^* \omega && [\partial([0,1]\times M) = \{0,1\}\times M] \\
&= \int_{[0,1]\times M} \mathrm{d}(\iota^* H^* \omega) && \text{[Stokes' theorem]} \\
&= \int_{[0,1]\times M} \iota^* H^* \mathrm{d}\omega && \text{[d and pullbacks commute]} \\
&= 0 && [\omega \text{ is a volume form, hence closed}]. \qquad \square
\end{aligned}
$$

Exercise 2.4 (Taken from [GN14]). Let M be a compact orientable manifold without boundary of dimension n, and let $\omega \in \Omega^{n-1}(M)$ be a form in M.

(a) Show that $\mathrm{d}\omega$ is not a volume form.

(b) Show that there does not exist an immersion $f\colon S^1 \longrightarrow \mathbb{R}$.

Solution. (a): Assume by contradiction that $\mathrm{d}\omega$ is a volume form.

$$
\begin{aligned}
0 &< \operatorname{vol}(M) \\
&= \int_M \mathrm{d}\omega && \text{[definition of volume]} \\
&= \int_{\partial M} \omega && \text{[Stokes' theorem]} \\
&= \int_{\varnothing} \omega && [\partial M = \varnothing] \\
&= 0.
\end{aligned}
$$

Contradiction.

(b): Assume by contradiction that there exists an immersion $f\colon S^1 \longrightarrow \mathbb{R}$. Consider the 1-form $\mathrm{d}f \in \Omega^1(S^1)$. Since f is an immersion, $\mathrm{d}f$ is a volume form on S^1. This contradicts (a). $\square$

Exercise 2.5 (divergence, taken from [GN14]). Let M be a compact manifold with boundary ∂M and let $\omega \in \Omega^n(M)$ be a volume form on M. The **divergence** of a vector field $X \in \mathfrak{X}(M)$ is the unique function $\operatorname{div}(X) \in C^\infty(M,\mathbb{R})$ which satisfies $L_X\omega = (\operatorname{div}(X))\omega$. Show that $\int_M \operatorname{div}(X)\omega = \int_{\partial M} \iota_X\omega$.

Solution.

$$
\begin{aligned}
\int_M \operatorname{div}(X)\omega &= \int_M L_X\omega && \text{[definition of divergence]} \\
&= \int_M \mathrm{d}\iota_X\omega + \int_M \iota_X \mathrm{d}\omega && \text{[Cartan's magic formula]} \\
&= \int_M \mathrm{d}\iota_X\omega && [\omega \text{ is of top degree} \Longrightarrow \mathrm{d}\omega = 0] \\
&= \int_{\partial M} \iota_X\omega && \text{[Stokes' theorem]}. \qquad \square
\end{aligned}
$$

3 Exercise sheet No. 3 - 26-11-2020

Exercise 3.1. Let V be an n-dimensional vector space. A linear map $\pi : V \to V$ is called **projection** if $\pi \circ \pi = \pi$. If $\langle \cdot, \cdot \rangle$ is an inner-product on V then π is called **orthogonal projection** if π is a projection and $\langle \pi(v), w\rangle = \langle v, \pi(w)\rangle$, for all $v, w \in V$. Using any isomorphism $\varphi : V \to \mathbb{R}^n$ we see that V has the structure of a smooth manifold with one chart (V, φ). With the inner product we may define a Riemannian metric as follows: for $p \in V$ and $v, w \in T_pV \cong V$ we set $g(p)(v, w) := \langle v, w\rangle$. Also $\pi(V) \subset V$ has the structure of a Riemannian manifold with the Riemannian metric induced from V. Show that $\pi : V \to \pi(V)$ is a Riemannian submersion.

Solution. First we show that π is a submersion. For $p \in V$ we have

$$\begin{aligned} d\pi(p) : T_pV \cong V \to T_{\pi(p)}(\pi(V)) \cong \pi(V), \\ v \mapsto \pi(v). \end{aligned}$$

This map is obviously surjective. Moreover, $\ker(d\pi(p)) = \ker(\pi)$ and $\ker(d\pi(p))^\perp = \ker(\pi)^\perp$. Now we show that

$$d\pi(p) : \ker(\pi)^\perp \to \pi(V)$$

is an isometry i.e. $g(\pi(p))(d\pi(p)v, d\pi(p)w) = g(p)(v, w)$ for all $v, w \in \ker(d\pi(p))^\perp$. More precisely, we show that $\langle \pi(v), \pi(w)\rangle = \langle v, w\rangle$, for all $v, w \in \ker(\pi)^\perp$. The proof is the following computation:

$$\begin{aligned} \langle \pi(v), \pi(w)\rangle &= \langle \pi\pi(v), w\rangle \quad &&[\pi \text{ is an orthogonal projection}] \\ &= \langle \pi(v), w\rangle &&[\pi \text{ is a projection}] \\ &= \langle v, w\rangle &&[v - \pi(v) \in \ker \pi \text{ and } v \in \ker \pi^\perp]. \end{aligned}$$ $\square$

Exercise 3.2 (Hopf fibration)**.** Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ and $S^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 = 1\}$ then the Hopf fibration is given by $\pi_{\mathrm{H}} : S^3 \to S^2$, $\pi_{\mathrm{H}}(z_1, z_2) = (2z_1\overline{z}_2, |z_1|^2 - |z_2|^2)$. Since $\mathbb{C}^2 \cong \mathbb{R}^4$ and $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ show that $\pi_{\mathrm{H}} : (S^3, g_3) \to (S^2, (1/4)g_2)$ is a Riemannian submersion.

Solution. Consider the following parametrization for S^3. In polar coordinates we have $z_1 = r_1e^{i\varphi_1}$ and $z_2 = r_2e^{i\varphi_2}$ where $r_1, r_2 \geq 0$ and $\varphi_1, \varphi_2 \in [0, 2\pi)$. Then

$$\begin{aligned} S^3 &= \left\{(r_1e^{i\varphi_1}, r_2e^{i\varphi_2}) \mid r_1, r_2 \geq 0, r_1^2 + r_2^2 = 1, \varphi_1, \varphi_2 \in [0, 2\pi)\right\} \\ &= \left\{(\cos(\psi)e^{i\varphi_1}, \sin(\psi)e^{i\varphi_2}) \mid \psi \in \left[0, \frac{\pi}{2}\right], \varphi_1, \varphi_2 \in [0, 2\pi)\right\} \\ &= \left\{(\psi, \varphi_1, \varphi_2) \mid \psi \in \left[0, \frac{\pi}{2}\right], \varphi_1, \varphi_2 \in [0, 2\pi)\right\}. \end{aligned}$$

Consider the following parametrization for S^2. In polar coordinates we have $z = re^{i\eta}$, where $r \geq 0$ and $\eta \in [0, 2\pi)$. Then

$$\begin{aligned} S^2 &= \left\{(re^{i\eta}, x) \mid x \in \mathbb{R}, r \geq 0, x^2 + r^2 = 1, \eta \in [0, 2\pi)\right\} \\ &= \left\{(\sin(\xi)e^{i\eta}, \cos(\xi)) \mid \xi \in [0, \pi], \eta \in [0, 2\pi)\right\} \end{aligned}$$

$$= \{(\xi,\eta) \mid \xi \in [0,\pi], \eta \in [0,2\pi)\}.$$

Then $\pi_{\mathrm{H}}(\cos(\psi)e^{i\varphi_1}, \sin(\psi)e^{i\varphi_2}) = (\sin(2\psi)e^{i(\varphi_1-\varphi_2)}, \cos(2\psi))$ and hence in the new coordinates $\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2) = (2\psi, \varphi_1 - \varphi_2 \mod 2\pi)$. Then the derivative is

$$d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2.$$

This is obviously a surjection. Its kernel is $\ker(d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2)) = \operatorname{span}\{e_2+e_3\} = \operatorname{span}\left\{\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right\}$. Now we compute the metrics g_3 and g_2 in the coordinates $(\psi,\varphi_1,\varphi_2)$ and (ξ,η), respectively. Since $S^3 \subset \mathbb{R}^4$ and $(x_1,...,x_4)$ are the standard coordinates on $\mathbb{R}^4$ we have

$$\begin{aligned} x_1 &= \cos(\psi)\cos(\varphi_1), \\ x_2 &= \cos(\psi)\sin(\varphi_1), \\ x_3 &= \sin(\psi)\cos(\varphi_2), \\ x_4 &= \sin(\psi)\sin(\varphi_2). \end{aligned}$$

Which implies

$$\begin{aligned} dx_1 &= -\sin(\psi)\cos(\varphi_1)d\psi - \cos(\psi)\sin(\varphi_1)d\varphi_1, \\ dx_2 &= -\sin(\psi)\sin(\varphi_1)d\psi + \cos(\psi)\cos(\varphi_1)d\varphi_1, \\ dx_3 &= \cos(\psi)\cos(\varphi_2)d\psi - \sin(\psi)\sin(\varphi_2)d\varphi_2, \\ dx_4 &= \cos(\psi)\sin(\varphi_2)d\psi + \sin(\psi)\cos(\varphi_2)d\varphi_2. \end{aligned}$$

Hence by computation we see

$$\begin{aligned} g_3 &= dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 + dx_4 \otimes dx_4 \\ &= d\psi \otimes d\psi + \cos^2(\psi)d\varphi_1 \otimes d\varphi_1 + \sin^2(\psi)d\varphi_2 \otimes d\varphi_2. \end{aligned}$$

Since $S^2 \subset \mathbb{R}^3$ and (y_1,y_2,y_3) are the standard coordinates on $\mathbb{R}^3$ we have

$$\begin{aligned} y_1 &= \sin(\xi)\cos(\eta), \\ y_2 &= \sin(\xi)\sin(\eta), \\ y_3 &= \cos(\eta). \end{aligned}$$

Hence

$$\begin{aligned} dy_1 &= \cos(\xi)\cos(\eta)d\xi - \sin(\xi)\sin(\eta)d\eta, \\ dy_2 &= \cos(\xi)\sin(\eta)d\xi + \sin(\xi)\cos(\eta)d\eta, \\ dy_3 &= -\sin(\xi)d\xi. \end{aligned}$$

Hence, computation yields

$$\begin{aligned} g_2 &= dy_1 \otimes dy_1 + dy_2 \otimes dy_2 + dy_3 \otimes dy_3 \\ &= d\xi \otimes d\xi + \sin^2(\xi)d\eta \otimes d\eta. \end{aligned}$$

Now we compute $\ker(d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2))^\perp$. Let $v = a\frac{\partial}{\partial\psi} + b\frac{\partial}{\partial\varphi_1} + c\frac{\partial}{\partial\varphi_2}$ then the condition $g_3\left(v, \frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right) = 0$ gives the equation

$$\cos^2(\psi)b + \sin^2(\psi)c = 0.$$

Assume now that $\psi \neq \frac{\pi}{2}$. Then $b = -\tan^2(\psi)c$ and hence

$$\begin{aligned}
\ker(d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2))^\perp &= \operatorname{span}\left\{\frac{\partial}{\partial\psi}, -\tan^2(\psi)\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right\} \\
&= \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-\tan^2(\psi)\\1\end{pmatrix}\right\}.
\end{aligned}$$

Let now $v, w \in \ker(d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2))^\perp$, i.e.

$$\begin{aligned}
v &= v_1\frac{\partial}{\partial\psi} + v_2\left(-\tan^2(\psi)\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right), \\
w &= w_1\frac{\partial}{\partial\psi} + w_2\left(-\tan^2(\psi)\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right).
\end{aligned}$$

Then

$$g_3(v,w) = v_1w_1 + v_2w_2\tan^2(\psi).$$

On the other hand we have

$$\begin{aligned}
&\frac{1}{4}g_2(d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2)v, d\pi_{\mathrm{H}}(\psi,\varphi_1,\varphi_2)w) \\
&\quad= \frac{1}{4}g_2\left(2v_1\frac{\partial}{\partial\xi} - v_2\left(\tan^2(\psi)+1\right)\frac{\partial}{\partial\eta}, 2w_1\frac{\partial}{\partial\xi} - w_2\left(\tan^2(\psi)+1\right)\frac{\partial}{\partial\eta}\right) \\
&\quad= \frac{1}{4}\left(4v_1w_1 + \sin^2(2\psi)v_2w_2\left(\tan^2(\psi)+1\right)^2\right) \\
&\quad= g_3(v,w).
\end{aligned}$$

□

Exercise 3.3 (metric on quotient by Lie group)**.** Let G be a Lie group, (M,g) be a Riemannian manifold and $G \times M \longrightarrow M$ be a free and proper action of G on M by isometries. So, we can form the quotient manifold M/G, which comes with a projection $\pi\colon M \longrightarrow M/G$ which is a surjective submersion. Show that there exists a unique Riemannian metric h on M/G such that π is a Riemannian submersion.

Solution. We prove uniqueness. For $p \in M$, note that $T_pM = \ker \mathrm{D}\pi(p) \oplus \ker \mathrm{D}\pi(p)^\perp$ and consider the following commutative diagram:

$$\begin{tikzcd}
\ker \mathrm{D}\pi(p) \oplus \ker \mathrm{D}\pi(p)^\perp \arrow[drr, "\mathrm{D}\pi(p)"] & & \ker \mathrm{D}\pi(p)^\perp \arrow[ll, hook', "\iota_p^\perp"'] \arrow[d, "\cong"', "\mathrm{D}\pi(p)\circ\iota_p^\perp"] \\
\ker \mathrm{D}\pi(p) \arrow[u, hook, "\iota_p"] \arrow[rr, "0"'] & & T_{\pi(p)}M/G
\end{tikzcd}.$$

π being a Riemannian submersion implies that $\mathrm{D}\pi(p) \circ \iota_p^{\perp} \colon \ker \mathrm{D}\pi(p)^{\perp} \longrightarrow T_{\pi(p)}M/G$ is an isometry, and this condition uniquely determines $h_{\pi(p)}$.

We prove existence. For each $p \in M$, define an inner product $h_{\pi(p)}$ on $T_{\pi(p)}M/G$ by requiring that $\mathrm{D}\pi(p) \circ \iota_p^{\perp} \colon \ker \mathrm{D}\pi(p)^{\perp} \longrightarrow T_{\pi(p)}M/G$ be an isometry. We need to check that $\pi(p) = \pi(q)$ implies $h(p) = h(q)$. By definition of π, $\pi(p) = \pi(q)$ is equivalent to there existing an $a \in G$ such that $q = ap$. We also denote by a the induced map $a \colon M \longrightarrow M$ by $a \in G$ and the action. By definition of π, we have that $\pi \circ a = \pi$ and $\mathrm{D}a(p) \colon T_pM \longrightarrow T_qM$ maps $\ker \mathrm{D}\pi(p)$ to $\ker \mathrm{D}\pi(q)$. Since a is an isometry, $\mathrm{D}a(p)$ maps $\ker \mathrm{D}\pi(p)^{\perp}$ to $\ker \mathrm{D}\pi(q)^{\perp}$. We now write a commutative diagram like the one before, but relating the tangent spaces at p and q:

$$\begin{tikzcd}
& \ker \mathrm{D}\pi(p) \oplus \ker \mathrm{D}\pi(p)^{\perp} \arrow[dl, "\mathrm{D}a(p)"'] & & \ker \mathrm{D}\pi(p)^{\perp} \arrow[ll, hook'] \arrow[dl, "\mathrm{D}a(p)"] \arrow[dd, "\mathrm{D}\pi(p)\circ\iota_p^{\perp}"] \\
\ker \mathrm{D}\pi(q) \oplus \ker \mathrm{D}\pi(q)^{\perp} & & \ker \mathrm{D}\pi(q)^{\perp} \arrow[ll, hook'] \arrow[dd, "\mathrm{D}\pi(q)\circ\iota_q^{\perp}"' near start] & \\
& \ker \mathrm{D}\pi(p) \arrow[uu, hook] \arrow[rr, "0"' near start] \arrow[dl, "\mathrm{D}a(p)"'] & & T_{\pi(p)}M/G \arrow[dl, "\mathrm{id}"] \\
\ker \mathrm{D}\pi(q) \arrow[uu, hook] \arrow[rr, "0"'] & & T_{\pi(q)}M/G &
\end{tikzcd}.$$

This diagram is commutative by the discussion above. To show that $h_{\pi(p)} = h_{\pi(q)}$, it suffices to show that $\mathrm{id} \colon (T_{\pi(p)}M/G, h_{\pi(p)}) \longrightarrow (T_{\pi(q)}M/G, h_{\pi(q)})$ is an isometry:

$$\begin{aligned}
&\mathrm{id} \text{ is an isometry} \\
\Longleftrightarrow{} &\mathrm{id} \circ \mathrm{D}\pi(p) \circ \iota_p^{\perp} \text{ is an isometry} && [\pi(p) \circ \iota_p^{\perp} \text{ is an isometry}] \\
\Longleftrightarrow{} &\mathrm{D}\pi(q) \circ \iota_q^{\perp} \circ \mathrm{D}a(p) \text{ is an isometry} && [\text{the diagram commutes}] \\
\Longleftrightarrow{} &\text{true} && [\mathrm{D}a(p) \text{ and } \mathrm{D}\pi(q) \circ \iota_q^{\perp} \text{ are isometries}].
\end{aligned}$$

Therefore, all the inner products $h_{\pi(p)}$ for $p \in M$ assemble into a Riemannian metric h on M/G. By the discussion above, π is a Riemannian submersion. $\square$

Exercise 3.4 (isometries of $\mathbb{R}^n$ and S^n)**.**

(a) Consider $\mathrm{Isom}(\mathbb{R}^n, g_{\mathbb{R}^n}) = \{h \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n \mid h \text{ is a smooth diffeomorphism}, h^* g_{\mathbb{R}^n} = g_{\mathbb{R}^n}\}$.

(a.1) For $x, y \in \mathbb{R}^n$, define

$$\begin{aligned}
\mathcal{A} &= \{\alpha \colon [0,1] \longrightarrow \mathbb{R}^n \mid \alpha \text{ is of class } C^1, \alpha(0) = x, \alpha(1) = y\} \\
L(\alpha) &= \int_0^1 |\dot{\alpha}(\tau)| \mathrm{d}\tau.
\end{aligned}$$

Show that $\|x - y\| = \inf_{\alpha \in \mathcal{A}} L(\alpha)$.

(a.2) Let $h \in \mathrm{Isom}(\mathbb{R}^n, g_{\mathbb{R}^n})$. Show that h preserves length, i.e. $\|h(x) - h(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$.

(a.3) Show that h is of the form $h(x) = Ax+b$, for $A \in \mathcal{O}(n)$ and $b \in \mathbb{R}^n$. Conclude that $\mathrm{Isom}(\mathbb{R}^n, g_{\mathbb{R}^n}) = \{x \longmapsto Ax + b \mid A \in \mathcal{O}(n), b \in \mathbb{R}^n\}$.

(b) Consider $\mathrm{Isom}(S^n, g_{S^n}) = \{h\colon S^n \longrightarrow S^n \mid h \text{ is a smooth diffeomorphism}, h^* g_{S^n} = g_{S^n}\}$.

(b.1) For $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$, define

$$\mathcal{A} = \{\alpha\colon [0,1] \longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \mid \alpha \text{ is of class } C^1, \alpha(0) = x, \alpha(1) = y\}$$
$$L(\alpha) = \int_0^1 |\dot{\alpha}(\tau)| \mathrm{d}\tau.$$

Show that $\|x - y\| = \inf_{\alpha \in \mathcal{A}} L(\alpha)$.

(b.2) Let $h \in \mathrm{Isom}(S^n, g_{S^n})$. Define $\overline{h}\colon \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ via

$$\overline{h}(x) := \|x\| h\Big(\frac{x}{\|x\|}\Big).$$

Show that $\overline{h}\colon \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ is an isometry.

(b.3) Show that $\overline{h}$ preserves length, i.e. $\|\overline{h}(x) - \overline{h}(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$.

(b.4) Show that $\overline{h}\colon \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ can be extended to a continuous map $\overline{h}\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ such that $\|\overline{h}(x) - \overline{h}(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^{n+1}$.

(b.5) Show that h is of the form $h(x) = Ax$, for $A \in \mathcal{O}(n)$. Conclude that $\mathrm{Isom}(S^n, g_{S^n}) = \{x \longmapsto Ax \mid A \in \mathcal{O}(n)\}$.

Solution. (a.1): We show that $\|x - y\| \geq \inf_{\alpha \in \mathcal{A}} L(\alpha)$. For this, define $\gamma(t) = x + t(y - x)$. Then, $\gamma \in \mathcal{A}$ and $\|x - y\| = L(\gamma) \geq \inf_{\alpha \in \mathcal{A}} L(\alpha)$. We show that $\|x - y\| \leq \inf_{\alpha \in \mathcal{A}} L(\alpha)$. For this, it suffices to assume that $\gamma \in \mathcal{A}$ and to prove that $L(\gamma) \geq \|x - y\|$. The proof is the following computation:

$$\begin{aligned} L(\gamma) &= \int_0^1 \|\dot{\gamma}(\tau)\| d\tau \quad &&\text{[by definition of length]} \\ &\geq \left\| \int_0^1 \dot{\gamma}(\tau) d\tau \right\| \\ &= \|x - y\| &&\text{[fundamental theorem of calculus].} \end{aligned}$$

(a.2): Pick $x, y \in \mathbb{R}^n$ and let $\alpha : [0,1] \to \mathbb{R}^n$ be a curve such that $\alpha(0) = x$ and $\alpha(1) = y$. Then

$$L(\alpha) = \int_0^1 \|\dot{\alpha}(\tau)\| \, d\tau = \int_0^1 \|dh(\alpha(\tau))\dot{\alpha}(\tau)\| \, d\tau = \int_0^1 \left\| \frac{d}{d\tau}(h \circ \alpha)(\tau) \right\| d\tau = L(h \circ \alpha).$$

Set $\mathcal{B} = \{\beta\colon [0,1] \longrightarrow \mathbb{R}^n \mid \beta \text{ is of class } C^1, \beta(0) = h(x), \beta(1) = h(y)\}$ to be the set of curves joining $h(x)$ and $h(y)$. We claim that the map

$$\Gamma : \mathcal{A} \to \mathcal{B}, \alpha \mapsto h \circ \alpha.$$

is bijective. To see this first we check injectivity. Assume that $\Gamma(\alpha_1) = \Gamma(\alpha_2)$ i.e. $h(\alpha_1(t)) = h(\alpha_2(t))$, for all $t \in [0,1]$. Then $h^{-1}(h(\alpha_1(t))) = h^{-1}(h(\alpha_2(t)))$ which implies

$\alpha_1(t) = \alpha_2(t)$, for all $t \in [0,1]$. Now we check that Γ is surjective. Let $\beta \in \mathcal{B}$. Then, $h^{-1} \circ \beta \in \mathcal{A}$ and $\Gamma(h^{-1} \circ \beta) = \beta$. So we conclude that Γ is bijective. Hence

$$\|x - y\| = \inf_{\alpha \in \mathcal{A}} L(\alpha) = \inf_{\alpha \in \mathcal{A}} L(\Gamma(\alpha)) = \inf_{\beta \in \mathcal{B}} L(\beta) = \|h(x) - h(y)\|.$$

(a.3): Consider $\overline{h} := h - h(0)$. Then $\overline{h}$ is an isometry of $\mathbb{R}^n$ fixing 0. From step (a.2) we have that $\left\|\overline{h}(x)\right\| = \|x\|$ for all $x \in \mathbb{R}^n$. We also have that $\left\langle \overline{h}(x), \overline{h}(y) \right\rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. Indeed,

$$\begin{aligned}
\left\langle \overline{h}(x), \overline{h}(y) \right\rangle &= \frac{1}{2}\left(\left\|\overline{h}(x)\right\|^2 + \left\|\overline{h}(y)\right\|^2 - \left\|\overline{h}(y) - \overline{h}(x)\right\|^2 \right) \\
&= \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|y - x\|^2 \right) \\
&= \langle x, y \rangle .
\end{aligned}$$

Next we show that $\overline{h}$ is linear, i.e. $\overline{h}(x+y) = \overline{h}(x) + \overline{h}(y)$ and $\overline{h}(\lambda x) = \lambda \overline{h}(x)$ for all $x, y \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$. Let $e_1, ..., e_n$ be the standard Euclidean basis of $\mathbb{R}^n$. Then by the previous computation we have that $\overline{h}(e_1), ..., \overline{h}(e_n)$ is an orthonormal basis of $\mathbb{R}^n$. Hence for every $x \in \mathbb{R}^n$ we consider its representation in the basis $\overline{h}(e_1), ..., \overline{h}(e_n)$, i.e.

$$\overline{h}(x) = \sum_{i=1}^{n} \left\langle \overline{h}(x), \overline{h}(e_i) \right\rangle \overline{h}(e_i) \underbrace{=}_{\text{Step 3}} \sum_{i=1}^{n} \langle x, e_i \rangle \, \overline{h}(e_i).$$

Hence this implies that $\overline{h}$ is linear. Since $\overline{h}$ is linear and maps an orthonormal basis to an orthonormal basis we have that there exists a matrix $A \in \mathcal{O}(n)$ such that $\overline{h}(x) = Ax$ for all $x \in \mathbb{R}^n$. Hence $h(x) = Ax + b$, where $b = h(0)$. Since h was arbitrary, we conclude that $\mathrm{Isom}(\mathbb{R}^n, g_{\mathbb{R}^n}) = \{x \longmapsto Ax + b \mid A \in \mathcal{O}(n), b \in \mathbb{R}^n\}$.

(b.1): In the case where $0 \in \mathbb{R}^{n+1}$ does not lie on the segment $[x, y]$ we proceed as in (a.1). Assume now that $0 \in [x, y]$. By the same argument as before we have $\|x - y\| \leq \inf_{\alpha \in \mathcal{A}} L(\alpha)$. To show the other inequality we proceed as follows. Since 0 is in the line segment $[x, y]$, we may assume (after possibly switching the roles of x and y) that $y = \lambda x$ for some $\lambda < 0$. Let $\eta \in \mathbb{R}^{n+1} \backslash \{0\}$ be a unit vector orthogonal to $x - y$. We consider the sequence of curves in $\mathbb{R}^{n+1} \backslash \{0\}$ defined by

$$\gamma_N(t) = \begin{cases}
x + t\frac{\frac{x}{N} - x}{\frac{1}{2} - \frac{1}{N}}, & t \in \left[0, \frac{1}{2} - \frac{1}{N}\right] \\
\frac{1}{N}x + \left(t - \frac{1}{2} + \frac{1}{N}\right)(\eta - x), & t \in \left[\frac{1}{2} - \frac{1}{N}, \frac{1}{2}\right] \\
\frac{1}{N}\eta + \left(t - \frac{1}{2}\right)(y - \eta), & t \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{N}\right] \\
\frac{1}{N}y + \left(t - \frac{1}{2} - \frac{1}{N}\right)\frac{y - \frac{1}{N}y}{\frac{1}{2} - \frac{1}{N}}, & t \in \left[\frac{1}{2} + \frac{1}{N}, 1\right]
\end{cases}$$

Then $\gamma_N \in \mathcal{A}$ and

$$L(\gamma_N) = \|x\|\left(1 - \frac{1}{N}\right) + \|\eta - x\|\frac{1}{N} + \|y - \eta\|\frac{1}{N} + \|y\|\left(1 - \frac{1}{N}\right) \overset{N \to \infty}{\longrightarrow} \|x\| + \|y\|.$$

Also, note that

$$\|x\| + \|y\| = \|x - y\|$$

since $y = \lambda x$, where $\lambda < 0$.

(b.2): We show that $\overline{h}:\mathbb{R}^{n+1}\backslash\{0\}\to\mathbb{R}^{n+1}\backslash\{0\}$ is an isometry. It suffices to show that $\overline{h}$ is bijective and that for all $x\in\mathbb{R}^{n+1}\setminus\{0\}$, the linear map $d\overline{h}(x)$ is an isometry, because in this case $\overline{h}$ would be a bijective local diffeomorphism.

$\overline{h}$ is injective: Assume $\overline{h}(x_1)=\overline{h}(x_2)$. Then

$$\|x_1\|\, h\left(\frac{x_1}{\|x_1\|}\right)=\|x_2\|\, h\left(\frac{x_2}{\|x_2\|}\right)$$

which implies that $\|x_1\|=\|x_2\|$. Hence

$$h\left(\frac{x_1}{\|x_1\|}\right)=h\left(\frac{x_2}{\|x_2\|}\right).$$

Since h is an isometry we have that $x_1=x_2$.

$\overline{h}$ is surjective: Let $y\in\mathbb{R}^{n+1}\backslash\{0\}$. Then,

$$x=\|y\|\, h^{-1}\left(\frac{y}{\|y\|}\right)$$

is such that $h(x)=y$.

Next we compute $d\overline{h}(x)$. For a path $\gamma(t)$ in $\mathbb{R}^{n+1}$ such that $\gamma(0)=x$ and $\dot{\gamma}(0)=v\in T_x\mathbb{R}^{n+1}=\operatorname{span}\{x\}\oplus\operatorname{span}\{x\}^{\perp}=\operatorname{span}\{x\}\oplus T_{\frac{x}{\|x\|}}S^n$ we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\overline{h}(\gamma(t))&=d\overline{h}(x)v\\&=\frac{\langle x,v\rangle}{\|x\|}h\left(\frac{x}{\|x\|}\right)+\|x\|\, dh\left(\frac{x}{\|x\|}\right)\left[\frac{v}{\|x\|}-\frac{x\langle x,v\rangle}{\|x\|^3}\right].\end{aligned}$$

If $v\in\operatorname{span}\{x\}^{\perp}$ then

$$d\overline{h}(x)v=dh\left(\frac{x}{\|x\|}\right)v$$

and if $v\in\operatorname{span}\{x\}$ then $v=\lambda x$ for some $\lambda\in\mathbb{R}$ and

$$d\overline{h}(x)v=\lambda\|x\|\, h\left(\frac{x}{\|x\|}\right).$$

Finally, show that $d\overline{h}(x)$ is an isometry. For $v=v'\oplus\lambda x$ and $w=w'\oplus\mu x$ we have

$$\begin{aligned}&\left\langle d\overline{h}(x)v,d\overline{h}(x)w\right\rangle\\&\quad=\left\langle d\overline{h}(x)v'+\lambda d\overline{h}(x)x,d\overline{h}(x)w'+\mu d\overline{h}(x)x\right\rangle\\&\quad=\left\langle dh\left(\frac{x}{\|x\|}\right)v'+\lambda\|x\|\, h\left(\frac{x}{\|x\|}\right),dh\left(\frac{x}{\|x\|}\right)w'+\mu\|x\|\, h\left(\frac{x}{\|x\|}\right)\right\rangle\\&\quad=\left\langle dh\left(\frac{x}{\|x\|}\right)v',dh\left(\frac{x}{\|x\|}\right)w'\right\rangle+\mu\|x\|\left\langle dh\left(\frac{x}{\|x\|}\right)v',h\left(\frac{x}{\|x\|}\right)\right\rangle\\&\qquad+\lambda\|x\|\left\langle dh\left(\frac{x}{\|x\|}\right)w',h\left(\frac{x}{\|x\|}\right)\right\rangle+\lambda\mu\|x\|^2\left\langle h\left(\frac{x}{\|x\|}\right),h\left(\frac{x}{\|x\|}\right)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle v', w' \rangle + \lambda\mu \|x\|^2 \\
&= \langle v', w' \rangle + \lambda\mu \langle x, y \rangle \\
&= \langle v, w \rangle .
\end{aligned}
$$

(b.3): For $x, y \in \mathbb{R}^{n+1} \backslash \{0\}$ recall the set $\mathcal{A}$ from the previous step. Set now $\mathcal{B} = \left\{ \beta : [0,1] \to \mathbb{R}^{n+1} \backslash \{0\} \mid C^1 \text{ with } \beta(0) = \overline{h}(x), \beta(1) = \overline{h}(y) \right\}$. Then as in the previous exercise we have a bijection

$$
\Gamma : \mathcal{A} \to \mathcal{B}, \quad \alpha \mapsto \overline{h} \circ \alpha
$$

and for all $\alpha \in \mathcal{A}$ we have $L(\alpha) = L(\Gamma(\alpha))$. Thus, as in the previous exercise, we deduce

$$
\begin{aligned}
\|x - y\| &= \inf_{\alpha \in \mathcal{A}} L(\alpha) \\
&= \inf_{\alpha \in \mathcal{A}} L(\Gamma(\alpha)) \\
&= \inf_{\beta \in \mathcal{B}} L(\beta) \\
&= \left\| \overline{h}(x) - \overline{h}(y) \right\| .
\end{aligned}
$$

(b.4): $\overline{h}$ is continuous on $\mathbb{R}^{n+1} \backslash \{0\}$. To check continuity at 0 we proceed as follows. Let x_n be a sequence in $\mathbb{R}^{n+1}$ such that $x_n \to 0$ as $n \to \infty$. Then

$$
\overline{h}(x_n) = \|x_n\| \, h\left(\frac{x_n}{\|x_n\|} \right) \to 0
$$

as $n \to \infty$ since h is bounded. So $\overline{h}$ is continuous as a map $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$. It remains to show that $\|\overline{h}(x) - \overline{h}(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^{n+1}$. If $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$, then the result is true as we have proven before. If $x, y = 0$ then the result is also true because $\overline{h}(0) = 0$. If $y = 0$ and $x \neq 0$, then

$$
\left\| \overline{h}(x) - \overline{h}(y) \right\| = \left\| \overline{h}(x) \right\| = \left\| \|x\| \, h\left(\frac{x}{\|x\|} \right) \right\| = \|x - y\| .
$$

(b.5): Repeat the proof of (a.3). $\square$

4 Exercise sheet No. 4 - 03-12-2020

Exercise 4.1. Show that the stereographic projection is a conformal equivalence between $S^n_R\backslash\{N\}$ and $\mathbb{R}^n$.

Solution. Denote $\rho = \sigma^{-1} = (\xi, \tau) = \mathbb{R}^n \longrightarrow S^n_R \setminus \{N\}$. ρ is given by

$$\rho(u) = \Big(\frac{2R^2 u}{|u|^2 + R^2}, R\frac{|u|^2 - R^2}{|u|^2 + R^2}\Big).$$

Denote by $\iota\colon S^n_R \setminus \{N\} \longrightarrow \mathbb{R}^{n+1}$ the inclusion map and $\overline{\rho} = \iota\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ (which is given by the same formula as ρ). We want to show that $g_{S^n_R} = f\sigma^* g_{\mathbb{R}^n}$, for some positive function f on M. Consider the following sequence of equivalences:

$$\begin{aligned}
&\exists f \in C^\infty(M, \mathbb{R}^+)\colon g_{S^n_R} = f\sigma^* g_{\mathbb{R}^n}\\
&\quad\iff \exists f \in C^\infty(M, \mathbb{R}^+)\colon \rho^* g_{S^n_R} = f g_{\mathbb{R}^n}\\
&\quad\iff \exists f \in C^\infty(M, \mathbb{R}^+)\colon \rho^*\iota^* g_{\mathbb{R}^{n+1}} = f g_{\mathbb{R}^n}\\
&\quad\iff \exists f \in C^\infty(M, \mathbb{R}^+)\colon \overline{\rho}^* g_{\mathbb{R}^{n+1}} = f g_{\mathbb{R}^n}\\
&\quad\iff \exists f \in C^\infty(M, \mathbb{R}^+)\colon\\
&\qquad\quad \forall u \in \mathbb{R}^n\colon\\
&\qquad\quad \forall V, W \in T_u\mathbb{R}^n = \mathbb{R}^n\colon\\
&\qquad\qquad (\overline{\rho}^* g_{\mathbb{R}^{n+1}})_u(V, W) = f(g_{\mathbb{R}^n})_u(V, W)\\
&\quad\iff \exists f \in C^\infty(M, \mathbb{R}^+)\colon\\
&\qquad\quad \forall u \in \mathbb{R}^n\colon\\
&\qquad\quad \forall V, W \in T_u\mathbb{R}^n = \mathbb{R}^n\colon\\
&\qquad\qquad (g_{\mathbb{R}^{n+1}})_{\overline{\rho}(u)}(\mathrm{D}\overline{\rho}(u)V, \mathrm{D}\overline{\rho}(u)W) = f(g_{\mathbb{R}^n})_u(V, W).
\end{aligned}$$

Given this, we start by computing $\mathrm{D}\overline{\rho}(u)$. We know that $\mathrm{D}\overline{\rho}(u)V = \mathrm{D}(\xi, \tau)(u)V = (\mathrm{D}\xi(u)V, \mathrm{D}\tau(u)V)$. We compute $\mathrm{D}\xi(u)$. Since $\xi^i = \frac{2R^2u^i}{u^ku^k + R^2}$, it's derivative is

$$\begin{aligned}
\frac{\partial \xi^i}{\partial u^j} &= \frac{2R^2\delta^i_j(u^ku^k + R^2) - (\delta^k_j u^k + u^k\delta^k_j)2R^2u^i}{(u^ku^k + R^2)^2}\\
&= \frac{2R^2\delta^i_j(u^ku^k + R^2) - 4R^2u^iu^j}{(u^ku^k + R^2)^2}.
\end{aligned}$$

So,

$$\mathrm{D}\xi(u) = \Big[\frac{\partial \xi^i}{\partial u^j}\Big]_{i,j} = \frac{2}{(|u|^2 + R^2)^2}\Big(R^2(|u|^2 + R^2)I - 2R^2uu^T\Big).$$

We compute $\mathrm{D}\tau(u)$. Since $\tau = R\frac{u^ku^k - R^2}{u^ku^k + R^2}$, it's derivative is

$$\begin{aligned}
\frac{\partial \tau}{\partial u^i} &= R\frac{2u^i(u^ku^k + R^2) - 2u^i(u^ku^k - R^2)}{(u^ku^k + R^2)^2}\\
&= R\frac{2u^iu^ku^k + 2u^iR^2 - 2u^iu^ku^k + 2u^iR^2}{(u^ku^k + R^2)^2}
\end{aligned}$$

$$= R\frac{4u^iR^2}{(u^ku^k + R^2)^2}$$
$$= 4R^3\frac{u^i}{(u^ku^k + R^2)^2}.$$

So,

$$\mathrm{D}\tau(u) = \left[\frac{\tau}{u^i}\right]_i = \frac{4R^3}{(|u|^2 + R^2)^2}u^T.$$

We now compute $(g_{\mathbb{R}^{n+1}})_{\overline{\rho}(u)}\Big(\mathrm{D}\overline{\rho}(u)V, \mathrm{D}\overline{\rho}(u)W\Big)$.

$$\begin{aligned}
&(g_{\mathbb{R}^{n+1}})_{\overline{\rho}(u)}\Big(\mathrm{D}\overline{\rho}(u)V, \mathrm{D}\overline{\rho}(u)W\Big)\\
&\quad= \Big\langle\Big(\mathrm{D}\xi(u)V, \mathrm{D}\tau(u)V\Big), \Big(\mathrm{D}\xi(u)W, \mathrm{D}\tau(u)W\Big)\Big\rangle_{\mathbb{R}^{n+1}}\\
&\quad= \Big\langle\mathrm{D}\xi(u)V, \mathrm{D}\xi(u)W\Big\rangle_{\mathbb{R}^n} + \Big\langle\mathrm{D}\tau(u)V, \mathrm{D}\tau(u)W\Big\rangle_{\mathbb{R}}\\
&\quad= (\mathrm{D}\xi(u)V)^T\mathrm{D}\xi(u)W + (\mathrm{D}\tau(u)V)^T\mathrm{D}\tau(u)W\\
&\quad= V^T\left(\frac{2}{(|u|^2 + R^2)^2}\Big(R^2(|u|^2 + R^2)I - 2R^2uu^T\Big)\right)^2W\\
&\qquad + V^T\frac{4R^3}{(|u|^2 + R^2)^2}u\frac{4R^3}{(|u|^2 + R^2)^2}u^TW\\
&\quad= \frac{4}{(|u|^2 + R^2)^4}V^T\Big(R^2(|u|^2 + R^2)I - 2R^2uu^T\Big)^2W + \frac{16R^6}{(|u|^2 + R^2)^4}V^Tuu^TW\\
&\quad= \frac{4}{(|u|^2 + R^2)^4}V^T\Big(R^4(|u|^2 + R^2)^2I - 4R^4(|u|^2 + R^2)uu^T + 4R^4|u|^2uu^T\Big)W\\
&\qquad + \frac{16R^6}{(|u|^2 + R^2)^4}V^Tuu^TW\\
&\quad= \frac{4}{(|u|^2 + R^2)^4}V^T\Big(R^4(|u|^2 + R^2)^2I - 4R^6uu^T\Big)W + \frac{16R^6}{(|u|^2 + R^2)^4}V^Tuu^TW\\
&\quad= \frac{4}{(|u|^2 + R^2)^4}V^T\Big(R^4(|u|^2 + R^2)^2I\Big)W\\
&\quad= \frac{4R^4}{(|u|^2 + R^2)^2}V^TW\\
&\quad= \frac{4R^4}{(|u|^2 + R^2)^2}(g_{\mathbb{R}^n})_u(V, W).
\end{aligned}$$
□

Exercise 4.2 (naturality of Riemannian volume element)**.** Let $\varphi : M \to N$ be a diffeomorphism of orientable manifolds. Then for any metric g on N we have

$$\varphi^*\mathrm{Vol}_g = \epsilon\mathrm{Vol}_{\varphi^*g},$$

where $\epsilon \in \{\pm 1\}$ and corresponds to the case where φ is orientation preserving or orientation reversing.

Solution. Let $(U_\alpha, \varphi_\alpha)$ be a chart on M and (V_β, ψ_β) be a chart on N such that $\varphi(U_\alpha) \subset V_\beta$. On V_β we denote the coordinates by y^i while on U_α we denote the coordinates by x^j. Then the metric g is of the form

$$g = g_{ij}dy^i \otimes dy^j$$

and the volume form

$$\mathrm{Vol}_g = \sqrt{\det\left(g_{ij}(y)\right)}dy^1 \wedge \ldots \wedge dy^n.$$

Hence

$$\begin{aligned}
\varphi^*\mathrm{Vol}_g &= \sqrt{\det\left(g_{ij}(\varphi(x))\right)}\frac{\partial y^1}{\partial x^{i_1}}\cdots\frac{\partial y^n}{\partial x^{i_n}}dx^{i_1} \wedge \ldots \wedge dx^{i_n} \\
&= \sqrt{\det\left(g_{ij}(\varphi(x))\right)}\det\left(\frac{\partial y^j}{\partial x^i}\right)dx^1 \wedge \ldots \wedge dx^n.
\end{aligned}$$

On the other hand we have

$$h = \varphi^* g = h_{st}dx^s \otimes dx^t = g_{ij}(\varphi(x))\frac{\partial y^i}{\partial x^s}\frac{\partial y^j}{\partial x^t}dx^s \otimes dx^t.$$

Hence

$$\mathrm{Vol}_h = \sqrt{\det\left(g_{ij}(\varphi(x))\right)}\left|\det\left(\frac{\partial y^i}{\partial x^s}\right)\right|dx^1 \wedge \ldots \wedge dx^n.$$

As a result, we have that if φ is orientation preserving then $\det\left(\frac{\partial y^i}{\partial x^s}\right) > 0$ and we have $\varphi^*\mathrm{Vol}_g = \mathrm{Vol}_{\varphi^* g}$. While if φ is orientation reversing we have $\det\left(\frac{\partial y^i}{\partial x^s}\right) < 0$ and hence $\varphi^*\mathrm{Vol}_g = -\mathrm{Vol}_{\varphi^* g}$. □

Exercise 4.3 (bi-invariant volume element on Lie group)**.** Let G be a compact connected Lie group and g be a left-invariant Riemannian metric on G. Show that:

(a) Vol_g is a left-invariant form on G;

(b) For every $h \in G$, we have that $R_h^*\mathrm{Vol}_g$ is a left-invariant form on G;

(c) For every $h \in G$, we have that $R_h^*\mathrm{Vol}_g$ is a positively oriented form on G *(Hint: consider the map $G \longrightarrow \mathrm{GL}(\mathfrak{g})$ given by $h \longmapsto \mathrm{D}L_h(h^{-1})\mathrm{D}R_{h^{-1}}(e)$ and recall that $\mathrm{GL}(\mathfrak{g})$ has two connected components, one of matrices with positive determinant and one of matrices with negative determinant)*;

(d) For every $h \in G$, there exists a unique $\phi_h \in \mathbb{R}^+$ such that $\phi_h\mathrm{Vol}_g = R_h^*\mathrm{Vol}_g$;

(e) The map $\phi\colon G \longrightarrow \mathbb{R}^+$ is a Lie group homomorphism;

(f) Vol_g is a right-invariant form on G *(Hint: it suffices to show that $\phi(G) = \{1\}$)*.

Solution. (a): It suffices to assume that $h \in G$, $p \in G$ and to prove that $(L_h^*\mathrm{Vol}_g)|_p = \mathrm{Vol}_g|_p$. We claim that there exists $(U, x^1, \ldots, x^n)$ a coordinate neighbourhood of $L_h(p)$ on G such that $L_h^{-1}(U) \cap U = \varnothing$. To see this, let U' be a coordinate neighbourhood of $L_h(p)$ and V' be a coordinate neighbourhood of p such that $U' \cap V' = \varnothing$ (these exist because G is Hausdorff), and define $U = U' \cap L_h(V')$. Then U is as desired, and we can define a $(V, y^1, \ldots, y^n)$ a coordinate neighbourhood on G by $V = L_h^{-1}(U)$ and $y^j = x^j \circ L_h$. With respect to these coordinate neighbourhoods, the Riemannian metric g is written

$$g|_U = \sum_{i,j=1}^{n} g_{ij}^U \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

$$g|_V = \sum_{i,j=1}^{n} g_{ij}^V \mathrm{d}y^i \otimes \mathrm{d}y^j.$$

From these expressions and using the fact that g is left-invariant, we conclude that $g_{ij}^V = g_{ij}^U \circ L_h$:

$$\begin{aligned}
\sum_{i,j=1}^{n} g_{ij}^V \mathrm{d}y^i \otimes \mathrm{d}y^j &= g|_V \\
&= L_h^*(g|_U) \\
&= L_h^* \sum_{i,j=1}^{n} g_{ij}^U \mathrm{d}x^i \otimes \mathrm{d}x^j \\
&= \sum_{i,j=1}^{n} (g_{ij}^U \circ L_h) \mathrm{d}y^i \otimes \mathrm{d}y^j.
\end{aligned}$$

We now show that $(L_h^* \mathrm{Vol}_g)|_V = \mathrm{Vol}_g|_V$:

$$\begin{aligned}
(L_h^* \mathrm{Vol}_g)|_V &= L_h^*(\mathrm{Vol}_g|_U) \\
&= L_h^* \sqrt{\det(g_{ij}^U)} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \\
&= \sqrt{\det(g_{ij}^U \circ L_h)} \mathrm{d}(x^1 \circ L_h) \wedge \ldots \wedge \mathrm{d}(x^n \circ L_h) \\
&= \sqrt{\det(g_{ij}^V)} \mathrm{d}(y^1) \wedge \ldots \wedge \mathrm{d}(y^n) \\
&= \mathrm{Vol}_g|_V.
\end{aligned}$$

(b): It suffices to assume that $l \in G$ and to prove that $L_l^* R_h^* \mathrm{Vol}_g = R_h^* \mathrm{Vol}_g$.

$$\begin{aligned}
L_l^* R_h^* \mathrm{Vol}_g &= (R_h \circ L_l)^* \mathrm{Vol}_g \\
&= (L_l \circ R_h)^* \mathrm{Vol}_g \\
&= R_h^* L_l^* \mathrm{Vol}_g \\
&= R_h^* \mathrm{Vol}_g.
\end{aligned}$$

(c): Consider the map

$$\begin{aligned}
\Phi \colon G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\
h &\longmapsto \mathrm{D}L_h(h^{-1}) \mathrm{D}R_{h^{-1}}(e).
\end{aligned}$$

We claim that it suffices to show that for all $h \in G$ the map $\Phi_h \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is positively oriented. This is because by definition of orientation of $T_{h^{-1}}G$ the map $\mathrm{D}L_h(h^{-1})$ is positively oriented, therefore $\Phi_h \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is positively oriented if and only if $\mathrm{D}R_{h^{-1}}(e)$ is positively oriented.

Recall that $\mathrm{GL}(\mathfrak{g})$ has connected components $\det^{-1}(\mathbb{R}^+)$ and $\det^{-1}(\mathbb{R}^-)$. We need to show that $\Phi(G) \subset \det^{-1}(\mathbb{R}^+)$, because in this case every Φ_g is a matrix with positive determinant, hence preserves orientation. We know that either $\Phi(G) \subset \det^{-1}(\mathbb{R}^+)$ or $\Phi(G) \subset \det^{-1}(\mathbb{R}^-)$, because Φ is continuous and G is connected. Since $\Phi(e) = \mathrm{id}_{\mathfrak{g}} \in \det^{-1}(\mathbb{R}^+)$, we conclude that $\Phi(G) \subset \det^{-1}(\mathbb{R}^+)$.

(d): Uniqueness is obvious. We prove existence. We claim that there exists a unique $\phi_h \in \mathbb{R}^+$ such that $\phi_h \mathrm{Vol}_g|_e = (R_h^* \mathrm{Vol}_g)|_e$. This is because $\mathrm{Vol}_g|_e$ and $(R_h^* \mathrm{Vol}_g)|_e$ are nonzero elements of the one dimensional vector space $\bigwedge_{j=1}^n T_eG$, so they are multiples of each other. Since R_h is positively oriented, we also know that ϕ_h is a positive number.

(e): We show that ϕ is smooth. For this, it suffices to show that the map $h \longmapsto (R_h^*\mathrm{Vol}_g)|_e$ is smooth. Since for any $v_1, \ldots, v_n \in T_eG$ we have $(R_h^*\mathrm{Vol}_g)|_e(v_1, \ldots, v_n) = \mathrm{Vol}_g|_h(\mathrm{D}R_h(e)v_1, \ldots, \mathrm{D}R_h(e)v_n)$ and that Vol_g is smooth, it suffices to show that for any $v \in T_eG$ the map $h \longmapsto \mathrm{D}R_h(e)v$ is smooth. Let γ be a curve such that $\gamma(0) = e$ and $\dot\gamma(0) = v$. Then,

$$\mathrm{D}R_h(e)v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R_h(\gamma(s)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma(s)h.$$

By definition of Lie group, this expression is smooth in h.

We show that ϕ is a homomorphism.

$$\begin{aligned}
\phi_p\phi_q\mathrm{Vol}_g &= \phi_p R_q^*\mathrm{Vol}_g && [\text{definition of } \phi_q]\\
&= R_q^*\phi_p\mathrm{Vol}_g && [\text{pullbacks are linear}]\\
&= R_q^*R_p^*\mathrm{Vol}_g && [\text{definition of } \phi_p]\\
&= (R_p \circ R_q)^*\mathrm{Vol}_g && [\text{functoriality of pullbacks}]\\
&= R_{qp}^*\mathrm{Vol}_g && \\
&= \phi_{qp}\mathrm{Vol}_g && [\text{definition of } \phi_{qp}].
\end{aligned}$$

(f): Since ϕ is continuous and G is compact, $\phi(G) \subset \mathbb{R}^+$ is a compact subset. Also, the image of a group homomorphism is a subgroup so we conclude that the set $\phi(G)$ is compact and a subgroup of $\mathbb{R}^+$. We show that if K is a compact subgroup of $\mathbb{R}^+$, then $K = \{1\}$. For this, assume by contradiction that K has an element $\lambda \in \mathbb{R}^+ \setminus \{1\}$. Then, $\{\lambda^n\}_{n\in\mathbb{N}}$ is a subset of K because K is a subgroup. We claim that the sequence λ_n does not have a convergent subsequence. If $\lambda > 1$, then λ_n diverges to ∞, and if $\lambda < 1$ then λ_n "converges" to 0 which does not belong to $\mathbb{R}^+$. So the sequence λ_n does not have convergent subsequences. This contradicts the fact that K is compact. □

Exercise 4.4 (reparametrizations)**.** Let $\gamma : [a, b] \to M$ be a piecewise C^1 path with partition $a = a_1 < a_2 < \ldots < a_k = b$. Furthermore, assume that $\dot\gamma(t) \neq 0$ for all $t \in [a_i, a_{i+1}]$ for all $i = 1, \ldots, k-1$. Denote by $L = L([\gamma])$. Show that there exists a unique reparametrization $\tilde\gamma : [0, L] \to M$ with unit speed, wherever it exists.

Solution. First we prove the statement for C^1 paths (not piecewise). Let $\gamma : [a, b] \to M$ be a C^1 path such that $\dot\gamma(t) \neq 0$ for all $t \in [a, b]$. Set $L := L([\gamma])$ and consider the arc-length function

$$s(t) = \int_a^t \|\dot\gamma(\tau)\|_g \, d\tau, \quad t \in [a, b].$$

Then we have that $s(a) = 0$, $s(b) = L$ and s is of class C^1. Since $\dot\gamma(t) \neq 0$ for all $t \in [a, b]$ we see that

$$\dot s(t) = \|\dot\gamma(t)\|_g > 0$$

which implies that s is strictly increasing. Hence s is a strictly increasing homeomorphism and by the inverse function theorem we conclude that the inverse of s is C^1, too. Denote the inverse of s by $\varphi : [0, L] \to [a, b]$. Its derivative is

$$\dot\varphi(\tau) = \frac{1}{\dot s(\varphi(\tau))} = \frac{1}{\|\dot\gamma(\varphi(\tau))\|_g}.$$

Consider the curve $\tilde{\gamma} = \gamma \circ \varphi : [0, L] \to M$. Then

$$\left\|\dot{\tilde{\gamma}}(\tau)\right\|_g^2 = \|\dot{\gamma}(\varphi(\tau))\dot{\varphi}(\tau)\|_g^2 = \|\dot{\gamma}(\varphi(\tau))\|_g^2 \frac{1}{\|\dot{\gamma}(\varphi(\tau))\|_g^2} = 1.$$

Now for piecewise C^1 paths. Let now $\gamma : [a, b] \to M$ be a piecewise C^1 path with a partition $a = a_1 < a_2 < ... < a_k = b$ such that $\gamma|_{[a_i, a_{i+1}]}$ is of class C^1 for all $i = 1, ..., k-1$ and such that $\dot{\gamma}(t) \neq 0$ for all $t \in [a_i, a_{i+1}]$ for all $i = 1, ..., k-1$. Denote $L_1 = L(\gamma|_{[a_1,a_2]})$, $L_2 = L(\gamma|_{[a_2,a_3]})$, ..., $L_{k-1} = L(\gamma|_{[a_{k-1},a_k]})$ and $\ell_1 = L_1$, $\ell_2 = L_1 + L_2$, ..., $\ell_{k-1} = \sum_{i=1}^{k-1} L_i$. As before, consider the functions

$$\begin{aligned} &s_1 : [a_1, a_2] \to [0, \ell_1], \quad s_1(t) = \int_a^t \|\dot{\gamma}(\tau)\|_g \, d\tau, \\ &s_i : [a_i, a_{i+1}] \to [\ell_{i-1}, \ell_i], \quad s_i(t) = \ell_{i-1} + \int_{a_i}^t \|\dot{\gamma}(\tau)\|_g \, d\tau, \end{aligned}$$

for all $i = 2, ..., k-1$ and

$$s(t) = \begin{cases} s_1(t), & t \in [a_1, a_2], \\ s_2(t), & t \in [a_2, a_3], \\ s_3(t), & t \in [a_3, a_4], \\ \vdots & \\ s_{k-1}(t), & t \in [a_{k-1}, a_k]. \end{cases}$$

We wish to show that s is a strictly increasing homeomorphism such that $s|_{[a_i,a_{i+1}]}$ is a C^1-diffeomorphism. It follows immediately that s is strictly increasing, continuous and $s|_{[a_i,a_{i+1}]}$ is a C^1-diffeomorphism for all $i = 1, ..., k-1$. As in the non-piecewise case denote for every $i = 1, ..., k-1$, $\varphi_1 : [0, \ell_1] \to [a_1, a_2]$ and $\varphi_i : [\ell_{i-1}, \ell_i] \to [a_i, a_{i+1}]$ the inverse of s_1 and s_i, respectively. Then φ_i is a C^1-diffeomorphism. Define

$$\varphi(\tau) = \begin{cases} \varphi_1(\tau), & \tau \in [0, \ell_1], \\ \varphi_2(\tau), & \tau \in [\ell_1, \ell_2], \\ \varphi_3(\tau), & \tau \in [\ell_2, \ell_3], \\ \vdots & \\ \varphi_{k-1}(\tau), & \tau \in [\ell_{k-2}, \ell_{k-1}]. \end{cases}$$

Then φ is continuous, φ is the inverse of s and $\varphi|_{[0,\ell_1]}$ and $\varphi|_{[\ell_i,\ell_{i+1}]}$ is a C^1-diffeomorphism. As in the non-piecewise case $\gamma \circ \varphi$ is the desired parametrized curve. $\square$

5 Exercise sheet No. 5 - 10-12-2020

Exercise 5.1 (Riemannian manifolds are metric spaces)**.** Let (M, g) be a Riemannian manifold. Show that the function $d_g \colon M \times M \longrightarrow \mathbb{R}_0^+$ given by

$$d_g(p, q) = \inf\{L(\gamma) \mid \gamma \colon [0, 1] \longrightarrow M \text{ is a piecewise } C^1 \text{ path with } \gamma(0) = p, \gamma(1) = q\}$$

is a metric on M, i.e. that it satisfies

(a) $d_g(p, q) = 0 \Leftrightarrow p = q$;

(b) $d_g(p, q) = d_g(q, p)$;

(c) $d_g(p, q) \leq d_g(r, p) + d_g(q, r)$.

Solution. (a): We show that $d_g(p, p) = 0$. For this, let ρ be the constant curve at p. Then, $d_g(p, p) = \inf_\gamma L(\gamma) \leq L(\rho) = 0$.

We show that $p \neq q$ implies $d_g(p, q) > 0$. For this, it suffices to show that there exists a constant $D > 0$ such that for every γ a piecewise smooth curve from p to q we have that $L(\gamma) \geq D > 0$. Choose a coordinate chart $\phi \colon U \subset M \longrightarrow U' \subset \mathbb{R}^n$ centred at p such that $q \notin U$. Then, we can write the Riemannian metric g with respect to this coordinate chart: for $x \in \mathbb{R}^n$, and $u, v \in T_x\mathbb{R}^n = \mathbb{R}^n$,

$$((\phi^{-1})^* g)_x(u, v) = \langle u, A(x)v\rangle_{\mathbb{R}^n}$$

for some matrix $A(x)$ which is symmetric and positive definite. Choose $r > 0$ such that the closed unit ball $\overline{B_r(0)}$ is contained in U'. Define

$$C = \min\{\langle u, A(x)u\rangle \mid u \in \mathbb{R}^n, \|u\| = 1, x \in \overline{B_r(0)}\}.$$

C is well defined because the set over which we are taking the minimum is compact, and $C > 0$ because $A(x)$ is positive definite. We claim that $D \coloneqq C^{1/2} r$ is the desired constant. To see this, we assume that γ is a curve from p to q and we must show that $L(\gamma) \geq D$. Choose $a \in \mathbb{R}^+$ such that $\gamma([0, a]) \subset \phi^{-1}(\overline{B_r(0)})$ and $\gamma(a) \in \phi^{-1}(S_r(0)) = \phi^{-1}(\partial \overline{B_r(0)})$. Define $c \colon [0, a] \longrightarrow \mathbb{R}^n$ by $c = \phi \circ \gamma$. Then,

$$\begin{aligned}
L(\gamma) &= \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{1/2} \mathrm{d}t \\
&\geq \int_0^a g(\dot\gamma(t), \dot\gamma(t))^{1/2} \mathrm{d}t \\
&= \int_0^a \langle \dot c(t), A(x)\dot c(t)\rangle_{\mathbb{R}^n}^{1/2} \mathrm{d}t \\
&\geq \int_0^a C^{1/2} \|\dot c(t)\|_{\mathbb{R}^n} \mathrm{d}t \\
&\geq C^{1/2} \left\| \int_0^a \dot c(t) \mathrm{d}t \right\| \\
&= C^{1/2} \|c(a) - c(0)\|_{\mathbb{R}^n} \\
&= C^{1/2} r.
\end{aligned}$$

(b): We show that d_g is symmetric. Define

$$\mathcal{A} = \left\{ \alpha : [0, 1] \xrightarrow{C^1} M \mid \alpha(0) = p, \alpha(1) = q \right\},$$

$$\mathcal{B} = \left\{ \beta : [0,1] \xrightarrow{C^1} M \mid \beta(0) = q, \beta(1) = p \right\}.$$

Then

$$\begin{aligned} \Gamma : \mathcal{A} &\to \mathcal{B} \\ \alpha &\mapsto (t \mapsto \alpha(1-t)) \end{aligned}$$

is a bijection and $L(\alpha) = L(\Gamma(\alpha))$, $\forall \alpha \in \mathcal{A}$. Hence

$$\begin{aligned} d_g(p,q) &= \inf_{\alpha \in \mathcal{A}} L(\alpha) \\ &= \inf_{\alpha \in \mathcal{A}} L(\Gamma(\alpha)) \\ &= \inf_{\beta \in \mathcal{B}} L(\beta) \\ &= d_g(q,p). \end{aligned}$$

(c): We show that d_g satisfies the triangle inequality. Let γ_1 be a C^1 path joining p and r and let γ_2 be a C^1 path joining r and q. Then $\gamma := \gamma_1 \circ \gamma_2$ which is defined by

$$\gamma(t) = \begin{cases} \gamma_1(2t), & t \in \left[0, \frac{1}{2}\right] \\ \gamma_2(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is a piecewise C^1 path joining p and q and one can check that

$$L(\gamma) = L(\gamma_1) + L(\gamma_2).$$

Then we have that

$$d_g(p,q) \leq L(\gamma_1) + L(\gamma_2).$$

Since γ_1 and γ_2 were arbitrary path's we take the infimum over all γ_1 joining p and r and we obtain

$$d_g(p,q) \leq d_g(p,r) + L(\gamma_2).$$

Taking now the infimum over all γ_2 joining r and q we obtain the triangle inequality. □

Exercise 5.2 (connection on $\mathbb{R}^n$)**.** On $\mathbb{R}^n$ we have the Euclidean connection which is defined as follows. For vector fields $X = (X^1, \dots, X^n)$ and $Y = (Y^1, \dots, Y^n)$ on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$ we define

$$\left(\nabla_X^{\mathbb{R}^n} Y\right)(p) = \left(X^i(p) \frac{\partial Y^1}{\partial x^i}(p), \dots, X^i(p) \frac{\partial Y^n}{\partial x^i}(p) \right).$$

Show that $\nabla^{\mathbb{R}^n}$ is a connection and that for vector fields $X, Y, Z \in C^\infty(T\mathbb{R}^n)$ we have

$$\nabla_X g_{\mathbb{R}^n}(Y,Z) = g_{\mathbb{R}^n}(\nabla_X Y, Z) + g_{\mathbb{R}^n}(Y, \nabla_X Z)$$

and

$$\nabla_X^{\mathbb{R}^n} Y - \nabla_Y^{\mathbb{R}^n} X = [X,Y].$$

Solution. Let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be two vector fields in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then

$$\left(\nabla^{\mathbb{R}^n}_{fX} Y\right)(p) = f(p)X^i(p)\frac{\partial Y^j}{\partial x^i}(p)\frac{\partial}{\partial x^j} = f(p)\left(\nabla^{\mathbb{R}^n}_{X} Y\right)(p),$$

and

$$\begin{aligned}\left(\nabla^{\mathbb{R}^n}_{X} (fY)\right)(p) &= X^i(p)\frac{\partial (fY^j)}{\partial x^i}(p)\frac{\partial}{\partial x^j} \\ &= X^i(p)\frac{\partial f}{\partial x^i}(p)Y^j(p)\frac{\partial}{\partial x^j} + f(p)X^i(p)\frac{\partial Y^j}{\partial x^i}(p)\frac{\partial}{\partial x^j} \\ &= X(f)(p)Y(p) + f(p)\left(\nabla^{\mathbb{R}^n}_{X} Y\right)(p).\end{aligned}$$

Moreover, its obvious that $\nabla^{\mathbb{R}^n}$ is $\mathbb{R}$-linear in both arguments. Hence $\nabla^{\mathbb{R}^n}$ is a connection on $\mathbb{R}^n$. Let now $Z = Z^k \frac{\partial}{\partial x^k}$ be another vector field on $\mathbb{R}^n$. Then

$$X(g_{\mathbb{R}^n}(Y,Z)) = X\left(\sum_{i=1}^n Y^i Z^i\right) = \sum_{i,k=1}^n \left[\left(\frac{\partial Y^i}{\partial x^k}\right) Z^i + Y^i \left(\frac{\partial Z^i}{\partial x^k}\right)\right] X^k,$$

and

$$\begin{aligned} g_{\mathbb{R}^n}(\nabla^{\mathbb{R}^n}_X Y, Z) &= g_{\mathbb{R}^n}\left(X^i \frac{\partial Y^k}{\partial x^i}\frac{\partial}{\partial x^k}, Z^j \frac{\partial}{\partial x^j}\right) \\ &= X^i Z^j \frac{\partial Y^k}{\partial x^i}\delta_{kj} \\ &= X^i Z^j \frac{\partial Y^j}{\partial x^i} \end{aligned}$$

and

$$\begin{aligned} g_{\mathbb{R}^n}(Y, \nabla^{\mathbb{R}^n}_X Z) &= g_{\mathbb{R}^n}\left(Y^j \frac{\partial}{\partial x^j}, X^i \frac{\partial Z^l}{\partial x^i}\frac{\partial}{\partial x^l}\right) \\ &= Y^j X^i \frac{\partial Z^l}{\partial x^i}\delta_{jl} \\ &= Y^j X^i \frac{\partial Z^j}{\partial x^i}. \end{aligned}$$

For the vector fields X and Y we have

$$[X,Y] = \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}.$$

Moreover,

$$\nabla^{\mathbb{R}^n}_X Y = X^j \frac{\partial Y^i}{\partial x^j}\frac{\partial}{\partial x^i}, \quad \text{and} \quad \nabla^{\mathbb{R}^n}_Y X = Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}.$$ □

Exercise 5.3 (extending functions and vector fields)**.** Let $\tilde{M}$ be a manifold and $M \subset \tilde{M}$ be an embedded submanifold. Denote by $\iota \colon M \longrightarrow \tilde{M}$ the inclusion map. Show that

(a) If $f \in C^\infty(M,\mathbb{R})$, then there exists $\tilde{f} \in C^\infty(\tilde{M},\mathbb{R})$ such that $\tilde{f} \circ \iota = f$.

(b) If $X \in \mathfrak{X}(M)$, then there exists $\tilde{X} \in \mathfrak{X}(\tilde{M})$ such that X is ι-related to $\tilde{X}$.

Solution. (a): Since $M \subset \tilde{M}$ is a submanifold, there exists a chart $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha, \tilde{V}_\alpha)$ on $\tilde{M}$ such that $\varphi_\alpha = \tilde{\varphi}_\alpha : U_\alpha = \tilde{U}_\alpha \cap M \to \tilde{V}_\alpha \cap (\mathbb{R}^k \times \underbrace{\{0\}}_{\in \mathbb{R}^{n-k}}) = V_\alpha$ is a diffeomorphism. Consider the function $f_\alpha = f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$, written as $f_\alpha = f_\alpha(x^1, ..., x^k)$. Consider the extension $f_\alpha^{\text{ex}} : \tilde{V}_\alpha \to \mathbb{R}$ given by $f_\alpha^{\text{ex}}(x^1, x^2, ..., x^n) = f_\alpha(x^1, ..., x^k)$. Then $\tilde{f}_\alpha : \tilde{U}_\alpha \to \mathbb{R}$ defined by $\tilde{f}_\alpha = f_\alpha^{\text{ex}} \circ \tilde{\varphi}_\alpha$ is an extension of f on U_α. Choose now a collection of charts $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)$ on $\tilde{M}$ that cover M and consider also $\tilde{U}_0 = \tilde{M} \backslash M$ is open. Then $\tilde{U}_0$, $\tilde{U}_\alpha$ is a cover of $\tilde{M}$ and we choose a partition of unity ρ_α, ρ_0 subordinate to the cover. Then $\tilde{f} = \rho_0 + \sum_\alpha \rho_\alpha \tilde{f}_\alpha$ is an extension of f.

(b): Since $M \subset \tilde{M}$ is a submanifold, there exists a chart $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha, \tilde{V}_\alpha)$ on $\tilde{M}$ such that $\varphi_\alpha = \tilde{\varphi}_\alpha : U_\alpha = \tilde{U}_\alpha \cap M \to \tilde{V}_\alpha \cap (\mathbb{R}^k \times \underbrace{\{0\}}_{\in \mathbb{R}^{n-k}}) = V_\alpha$ is a diffeomorphism. The vector field X in the coordinates given by the chart $(U_\alpha, \varphi_\alpha, V_\alpha)$ is of the form

$$X_\alpha(x) = d\varphi_\alpha(\varphi_\alpha^{-1}(x))X(\varphi_\alpha^{-1}(x)) = X^1(x)\frac{\partial}{\partial x^1} + ... + X^k(x)\frac{\partial}{\partial x^k},$$

where $x = (x^1, ..., x^k) \in V_\alpha$. Extend the functions $X^i(x)$ on all of $\tilde{V}_\alpha$ as in the first part and we obtain a vector field X_α^{ex} on $\tilde{V}_\alpha$. Hence we obtain a vector field $\tilde{X}_\alpha$ on $\tilde{U}_\alpha$ by

$$\tilde{X}_\alpha(p) = d\tilde{\varphi}_\alpha^{-1}(\tilde{\varphi}_\alpha(p))X_\alpha^{\text{ex}}(\tilde{\varphi}_\alpha(p)).$$

For $p \in U_\alpha$ we have

$$\begin{aligned}
\tilde{X}_\alpha(p) &= d\tilde{\varphi}_\alpha^{-1}(\tilde{\varphi}_\alpha(p))X_\alpha^{\text{ex}}(\tilde{\varphi}_\alpha(p)) \\
&= d\tilde{\varphi}_\alpha^{-1}(\tilde{\varphi}_\alpha(p))X_\alpha(\tilde{\varphi}_\alpha(p)) \\
&= d\tilde{\varphi}_\alpha^{-1}(\tilde{\varphi}_\alpha(p))d\varphi_\alpha(\varphi_\alpha^{-1}(\varphi_\alpha(p)))X(\varphi_\alpha^{-1}(\varphi_\alpha(p))) \\
&= d\tilde{\varphi}_\alpha^{-1}(\varphi_\alpha(p))d\varphi_\alpha(p)X(p) \\
&= d\varphi_\alpha^{-1}(\varphi_\alpha(p))d\varphi_\alpha(p)X(p) \\
&= X(p).
\end{aligned}$$

Choose now a collection of charts $(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)$ on $\tilde{M}$ that cover M and consider also $\tilde{U}_0 = \tilde{M} \backslash M$ is open. Then $\tilde{U}_0$, $\tilde{U}_\alpha$ is a cover of $\tilde{M}$ and we choose a partition of unity ρ_α, ρ_0 subordinate to the cover. Then $\tilde{X} = \sum_\alpha \rho_\alpha \tilde{X}_\alpha$ is an extension of X. □

Exercise 5.4 (connections are local)**.** Let M be a manifold and ∇ be an affine connection on M. Prove that $\nabla_X Y|_p$ only depends on X_p and on the values of Y along a curve tangent to X_p.

Solution. Let $(U, x^1, \ldots, x^n)$ be a coordinate neighbourhood of p. Recall that the affine connection can be given in coordinates by

$$\nabla_X Y = \sum_{i=1}^n \Big(X(Y^i) + \sum_{j,k=1}^n \Gamma_{jk}^i X^i Y^k \Big) \frac{\partial}{\partial x^i}.$$

At the point p, this expression becomes

$$\nabla_X Y|_p = \sum_{i=1}^n \Big(X_p(Y^i) + \sum_{j,k=1}^n \Gamma_{jk}^i(p) X_p^i Y_p^k \Big) \frac{\partial}{\partial x^i}\Big|_p.$$

Notice that if γ is a curve tangent to X at p, i.e. $\gamma(0) = p$ and $(\dot{\gamma})(0) = X_p$, then

$$X_p(Y^i) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Y^i(\gamma(t)).$$

This proves the result. $\square$

Exercise 5.5 (connection on submanifold)**.** Let N be a manifold, $M \subset N$ be an embedded submanifold, and $\iota\colon M \longrightarrow N$ be the inclusion map. Let g^N be a Riemannian metric on N and define $g^M = \iota^* g^N$, which is a Riemannian metric on M. For every $p \in M$, we can consider the orthogonal complement $T_pM^\perp$ of T_pM inside T_pN. We have the decomposition $T_pN = T_pM \oplus T_pM^\perp$. In other words, for every $v \in T_pN$ there exist unique $v^\top \in T_pM$ and $v^\perp \in T_pM^\perp$ such that $v = v^\top + v^\perp$. If ∇^N is an affine connection on N, define an affine connection ∇^M on M via

$$\nabla^M_X Y = \left(\nabla^N_{\tilde{X}} \tilde{Y}\right)^\top,$$

where $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$ are extensions of $X, Y \in \mathfrak{X}(M)$ to N. Show that

(a) ∇^M is well defined and a connection.

(b) If ∇^N is symmetric then ∇^M is symmetric.

(c) If ∇^N is compatible with g^N then ∇^M is compatible with g^M.

(d) If ∇^N is the Levi-Civita connection with respect to g^N then ∇^M is the Levi-Civita connection with respect to g^M.

Solution. (a): We show that ∇^M is well defined. This is true, because by exercise 5.3, the extensions $\tilde{X}, \tilde{Y}$ exist, and by exercise 5.4, the result is independent of the choice of extensions.

We show that ∇^M is a connection. ∇^M is C^∞-linear on the first variable:

$$\begin{aligned}
\nabla^M_{fX} Y|_p &= \left(\nabla^N_{\tilde{f}\tilde{X}} \tilde{Y}\right)^\top_p \\
&= \left(\tilde{f} \nabla^N_{\tilde{X}} \tilde{Y}\right)^\top_p \\
&= f(p) \left(\nabla^N_{\tilde{X}} \tilde{Y}\right)^\top_p \\
&= f(p) \nabla^M_X Y|_p.
\end{aligned}$$

∇^M satisfies the Leibniz rule on the second variable:

$$\begin{aligned}
\nabla^M_X fY|_p &= \left(\nabla^N_{\tilde{X}} \tilde{f}\tilde{Y}\right)^\top_p \\
&= \left(\tilde{X}(\tilde{f})\tilde{Y} + \tilde{f} \nabla^N_{\tilde{X}} \tilde{Y}\right)^\top_p \\
&= X(f)Y|_p + f(p) \left(\nabla^N_{\tilde{X}} \tilde{Y}\right)^\top_p \\
&= X(f)Y|_p + f(p) \nabla^M_X Y|_p.
\end{aligned}$$

(b): Notice that $[\tilde{X}, \tilde{Y}]_p = [X, Y]_p \in T_pM$ implies that $[\tilde{X}, \tilde{Y}]_p^\top = [\tilde{X}, \tilde{Y}]_p = [X, Y]_p$.

$$\nabla^M_X Y|_p - \nabla^M_Y X|_p = \left(\nabla^N_{\tilde{X}} \tilde{Y}\right)^\top_p - \left(\nabla^N_{\tilde{Y}} \tilde{X}\right)^\top_p$$

$$
\begin{aligned}
&= \left(\nabla^N_{\tilde{X}}\tilde{Y} - \nabla^N_{\tilde{Y}}\tilde{X}\right)^\top_p \\
&= [\tilde{X},\tilde{Y}]^\top_p \\
&= [\tilde{X},\tilde{Y}]_p \\
&= [X,Y]_p.
\end{aligned}
$$

(c):

$$
\begin{aligned}
\langle \nabla^M_X Y, Z\rangle^M_p + \langle Y, \nabla^M_X Z\rangle^M_p &= \langle (\nabla^N_{\tilde{X}}\tilde{Y})^\top_p, Z\rangle^M_p + \langle Y, (\nabla^N_{\tilde{X}}\tilde{Z})^\top_p\rangle^M_p \\
&= \langle (\nabla^N_{\tilde{X}}\tilde{Y})^\top_p, \tilde{Z}\rangle^N_p + \langle \tilde{Y}, (\nabla^N_{\tilde{X}}\tilde{Z})^\top_p\rangle^N_p \\
&= \langle (\nabla^N_{\tilde{X}}\tilde{Y})_p, \tilde{Z}\rangle^N_p + \langle \tilde{Y}, (\nabla^N_{\tilde{X}}\tilde{Z})_p\rangle^N_p \\
&= \tilde{X}\cdot\langle \tilde{Y},\tilde{Z}\rangle^N_p \\
&= X\cdot\langle Y,Z\rangle^M_p.
\end{aligned}
$$

(d): If ∇^N is the Levi-Civita connection with respect to g^N, then ∇^N is symmetric and compatible with g^N, therefore ∇^M is symmetric and compatible with g^M, therefore ∇^M is the Levi-Civita connection with respect to g^M. □

Exercise 5.6 (divergence)**.** Let M be an orientable smooth manifold M with volume form ω and with possibly nonempty boundary ∂M. The divergence of a vector field $X \in \mathfrak{X}$ is the unique function $\operatorname{div} X \in C^\infty(M,\mathbb{R})$ such that

$$\operatorname{div}(X)\omega = L_X\omega.$$

Assume in addition that (M,g) is Riemannian, compact and that ω is the Riemannian volume element. Then ∂M is Riemannian as well, so it has a Riemannian volume element which we denote by ϖ. Denote by N the outward unit normal vector field to ∂M (which is a vector field defined on an open neighbourhood U of ∂M).

(a) Prove the divergence theorem: for all $X \in \mathfrak{X}(M)$, we have that

$$\int_M \operatorname{div}(X)\omega = \int_{\partial M} g(X,N)\varpi.$$

(b) Prove the Leibniz rule for the divergence: for all $u \in C^\infty(M,\mathbb{R})$ and $X \in \mathfrak{X}(M)$, we have that

$$\operatorname{div}(uX) = u\operatorname{div}(X) + g(\nabla u, X).$$

(c) Prove the integration by parts formula: for all $u \in C^\infty(M,\mathbb{R})$ and $X \in \mathfrak{X}(M)$, we have that

$$\int_M g(\nabla u, X)\omega = -\int_M u\operatorname{div}(X)\omega + \int_{\partial M} ug(X,N)\varpi.$$

Solution. (a): We proved in exercise 2.5 from exercise sheet 2 that

$$\int_M \operatorname{div}(X)\omega = \int_{\partial M} \iota_X\omega.$$

It remains to show that $\iota_X \omega = g(X, N)\overline{\omega}$. For this, let $p \in M$ and $E_1, \ldots, E_n$ be an orthonormal basis of T_pM with $E_1 = N_p$. Then, $E_2, \ldots, E_n$ is an orthonormal basis of $T_p(\partial M)$. Denote by $E^1, \ldots, E^n$ the basis of T_p^*M dual to $E_1, \ldots, E_n$. Then,

$$\begin{aligned}
&(\iota_X\omega)_p(E_2, \ldots, E_n) \\
&\quad = \omega_p(X, E_2, \ldots, E_n) && \text{[definition of } \iota_X\text{]} \\
&\quad = E^1 \wedge \cdots \wedge E^n(X, E_2, \ldots, E_n) && \text{[def. Riemannian volume element]} \\
&\quad = E^1(X) && \text{[definition of } \wedge\text{]} \\
&\quad = g_p(E_1, X) && [E_1, \ldots, E_n \text{ is orthonormal}] \\
&\quad = g_p(N, X) && [E_1 = N] \\
&\quad = g_p(N, X)\overline{\omega}(E_2, \ldots, E_n) && \text{[def. Riemannian volume element].}
\end{aligned}$$

(b): Since

$$\begin{aligned}
\operatorname{div}(uX)\omega &= L_{uX}\omega && \text{[definition of divergence]} \\
&= \mathrm{d}\iota_{uX}\omega + \iota_{uX}\mathrm{d}\omega && \text{[Cartan's magic formula]} \\
&= \mathrm{d}\iota_{uX}\omega && [\omega \text{ is of top degree}] \\
&= \mathrm{d}(u\iota_X\omega) \\
&= \mathrm{d}u \wedge \iota_X\omega + u\mathrm{d}\iota_X\omega && \text{[Leibniz rule for d]} \\
&= \mathrm{d}u \wedge \iota_X\omega + uL_X\omega && \text{[Cartan's magic formula]} \\
&= \mathrm{d}u \wedge \iota_X\omega + u\operatorname{div}(X)\omega && \text{[definition of divergence],}
\end{aligned}$$

it suffices to show that $\mathrm{d}u \wedge \iota_X\omega = g(\nabla u, X)\omega$. This is true because

$$\begin{aligned}
0 &= \iota_X(\mathrm{d}u \wedge \omega) && [\mathrm{d}u \wedge \omega = 0 \text{ because } \omega \text{ is of top degree}] \\
&= (\iota_X\mathrm{d}u)\omega - \mathrm{d}u \wedge \iota_X\omega && \text{[Leibniz rule for } \iota_X\text{]} \\
&= \mathrm{d}u(X)\omega - \mathrm{d}u \wedge \iota_X\omega && \text{[definition of } \iota_X\text{]} \\
&= g(\nabla u, X)\omega - \mathrm{d}u \wedge \iota_X\omega && \text{[definition of gradient].}
\end{aligned}$$

(c):

$$\begin{aligned}
\int_M g(\nabla u, X)\omega &= -\int_M u\operatorname{div}(X)\omega + \int_M \operatorname{div}(uX)\omega && \text{[by (b)]} \\
&= -\int_M u\operatorname{div}(X)\omega + \int_{\partial M} ug(X, N)\overline{\omega} && \text{[by (a)].}
\end{aligned}$$

□

6 Exercise sheet No. 6 - 17-12-2020

Exercise 6.1 (Laplacian)**.** Let (M, g) be an orientable connected Riemannian manifold, possibly with nonempty boundary ∂M. Denote by ∇ the Levi-Civita connection on M, by ω the Riemannian volume element of M, and by $\overline{\omega}$ the Riemannian volume element of ∂M. For $u \in C^\infty(M, \mathbb{R})$, the **Laplacian** of u is a function Δu defined by

$$\Delta u = \operatorname{div}(\nabla u).$$

A function u is **harmonic** if $\Delta u = 0$.

(a) Prove Green's identities: if $u, v \in C^\infty(M)$, then

$$\int_M u\Delta v\omega + \int_M g(\nabla u, \nabla v)\omega = \int_{\partial M} uN(v)\overline{\omega},$$
$$\int_M (u\Delta v - v\Delta u)\,\omega = \int_{\partial M} (uN(v) - vN(u))\,\overline{\omega}.$$

(b) Show that if $\partial M \neq \varnothing$ and u and v are harmonic functions such that $u|_{\partial M} = v|_{\partial M}$, then $u \equiv v$.

(c) Show that if $\partial M = \varnothing$ and u is a harmonic function then u is constant.

Solution. (a): We prove the first of Green's identities:

$$\begin{aligned}
&\int_M g(\nabla u, \nabla v)\omega \\
&\quad = -\int_M u\operatorname{div}(\nabla v)\omega + \int_{\partial M} ug(\nabla v, N)\overline{\omega} && \text{[integration by parts for div with } X = \nabla v\text{]} \\
&\quad = -\int_M u\Delta v\omega + \int_{\partial M} udv(N)\overline{\omega} && \text{[definition of } \Delta,\ \nabla\text{]} \\
&\quad = -\int_M u\Delta v\omega + \int_{\partial M} uN(v)\overline{\omega}.
\end{aligned}$$

We prove the second of Green's identities:

$$\begin{aligned}
&\int_M (u\Delta v - u\Delta v)\omega \\
&\quad = -\int_M g(\nabla u, \nabla v)\omega + \int_{\partial M} uN(v)\overline{\omega} \quad \text{[by the first Green identity]} \\
&\qquad + \int_M g(\nabla v, \nabla u)\omega - \int_{\partial M} vN(u)\overline{\omega} \\
&\quad = \int_{\partial M} (uN(v) - vN(u))\overline{\omega}.
\end{aligned}$$

(b): Define $z = u - v$. Then $z|_{\partial M} = 0$ and z is harmonic. We want to show that $z = 0$.

$$\begin{aligned}
\int_M \|\nabla z\|^2\omega &= \int_M g(\nabla z, \nabla z)\omega \\
&= -\int_M z\Delta z\omega + \int_{\partial M} zN(z)\overline{\omega} && \text{[by Green's identities]} \\
&= 0. && [\Delta z = 0, z|_{\partial M} = 0]
\end{aligned}$$

Therefore, $\nabla z = 0$, which implies that $\mathrm{d}z = 0$ and that z is constant. Since $z|_{\partial M} = 0$ and $\partial M \neq \varnothing$, we conclude that z is constant equal to 0.

(c):

$$\begin{aligned}
\int_M \|\nabla u\|^2 \omega &= \int_M g(\nabla u, \nabla u)\omega \\
&= -\int_M u\Delta u\omega + \int_{\partial M} uN(u)\varpi && \text{[by Green's identities]} \\
&= 0. && [\Delta u = 0, \partial M = \varnothing]
\end{aligned}$$

Therefore, $\nabla u = 0$, which implies that $\mathrm{d}u = 0$ and that u is constant. □

Exercise 6.2 (Eigenvalues of the Laplacian)**.** Let (M, g) be a closed oriented Riemannian manifold, with Riemannian volume element ω. A real number λ is called **eigenvalue** of the Laplacian if there exists a $u \in C^\infty(M)\backslash\{0\}$ such that

$$\Delta u = \lambda u.$$

In this case the function u is called **eigenvector** (or **eigenfunction**) of Δ to the eigenvalue λ.

(a) Show that 0 is an eigenvalue of Δ and all other eigenvalues are strictly negative.

(b) Let λ and μ be two distinct eigenvalues of Δ, with corresponding eigenfunctions u and v. Show that
$$\int_M uv\omega = 0.$$

Solution. (a): We show that 0 is an eigenvalue of Δ. For this, it suffices to show that there exists a nonvanishing function u such that $\Delta u = 0$. $u = 1$ is such a function.

We show that if λ is an eigenvalue of Δ, then $\lambda \leq 0$. Since λ is an eigenvalue of Δ, there exists a nonvanishing function u such that $\Delta u = \lambda u$. By first of Green's identities with $v = u$,

$$\begin{aligned}
&\int_M u\Delta u\omega + \int_M g(\nabla u, \nabla u)\omega = \int_{\partial M} uN(u)\varpi \\
\iff &\lambda \int_M u^2\omega + \int_M g(\nabla u, \nabla u)\omega = 0 \\
\iff &\lambda = -\frac{\int_M g(\nabla u, \nabla u)\omega}{\int_M u^2\omega} \leq 0.
\end{aligned}$$

(b):

$$\begin{aligned}
(\mu - \lambda)\int_M uv\omega &= \int_M (u\Delta v - v\Delta u) && [\Delta u = \lambda u \text{ and } \Delta v = \mu v] \\
&= \int_{\partial M} (uN(v) - vN(u))\varpi && \text{[second Green identity]} \\
&= 0 && [\partial M = 0].
\end{aligned}$$

Since $\mu - \lambda = 0$, we conclude that $\int_M uv\omega = 0$. □

Exercise 6.3 (covariant derivative in coordinates)**.** Let M be a manifold, ∇ be a linear connection on M, ω a 1-form on M and X a vector field on M. Show that the coordinate expression of $\nabla_X\omega$ is

$$\nabla_X\omega = \left(X^i\frac{\partial\omega_k}{\partial x^i} - X^i\omega_j\Gamma^j_{ik}\right)dx^k,$$

where Γ^k_{ij} are the Christoffel-Symbols of ∇ on TM. Find a coordinate formula for $\nabla_X F$, where $F \in \mathcal{T}^k_l M$ is a tensor field of rank (k,l).

Solution. Let $X = X^i\frac{\partial}{\partial x^i}$, $\omega = \omega_k \mathrm{d}x^k$ and $F = F^{j_1\dots j_l}_{i_1\dots i_k}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}\otimes \mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}$. Then, $\nabla_{\frac{\partial}{\partial x^i}}\mathrm{d}x^k = -\Gamma^k_{ir}\mathrm{d}x^r$:

$$\begin{aligned}
\left(\nabla_{\frac{\partial}{\partial x^i}}\mathrm{d}x^k\right)\left(\frac{\partial}{\partial x^l}\right) &= \nabla_{\frac{\partial}{\partial x^i}}\left(\mathrm{d}x^k\left(\frac{\partial}{\partial x^l}\right)\right) - \mathrm{d}x^k\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}\right)\\
&= \nabla_{\frac{\partial}{\partial x^i}}\left(\delta^k_l\right) - \mathrm{d}x^k\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}\right)\\
&= -\mathrm{d}x^k\left(\Gamma^r_{il}\frac{\partial}{\partial x^r}\right)\\
&= -\Gamma^k_{il}\\
&= -\Gamma^k_{ir}\mathrm{d}x^r\left(\frac{\partial}{\partial x^l}\right).
\end{aligned}$$

We show that $\nabla_X\omega = X^i\left(\frac{\partial\omega_k}{\partial x^i} - \omega_j\Gamma^j_{ik}\right)\mathrm{d}x^k$:

$$\begin{aligned}
\nabla_X\omega &= \nabla_{X^i\frac{\partial}{\partial x^i}}\left(\omega_k\mathrm{d}x^k\right) && \text{[coordinate expressions for } X,\ \omega]\\
&= X^i\nabla_{\frac{\partial}{\partial x^i}}\left(\omega_k\mathrm{d}x^k\right) && [\nabla \text{ is } C^\infty\text{-linear in 1st variable}]\\
&= X^i\left(\left(\nabla_{\frac{\partial}{\partial x^i}}\omega_k\right)\mathrm{d}x^k + \omega_k\nabla_{\frac{\partial}{\partial x^i}}\mathrm{d}x^k\right) && [\nabla \text{ obeys Leibniz rule in 2nd variable}]\\
&= X^i\left(\frac{\partial\omega_k}{\partial x^i}\mathrm{d}x^k - \omega_k\Gamma^k_{ir}\mathrm{d}x^r\right) && \text{[by the computation above]}\\
&= X^i\left(\frac{\partial\omega_k}{\partial x^i}\mathrm{d}x^k - \omega_j\Gamma^j_{ik}\mathrm{d}x^k\right) && \text{[change names of the indices]}\\
&= X^i\left(\frac{\partial\omega_k}{\partial x^i} - \omega_j\Gamma^j_{ik}\right)dx^k.
\end{aligned}$$

We compute the local coordinate expression for $\nabla_X F$:

$$\begin{aligned}
&\nabla_X F\\
&= \nabla_{X^a\frac{\partial}{\partial x^a}}\left(F^{j_1\dots j_l}_{i_1\dots i_k}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}\right)\\
&= X^a\nabla_{\frac{\partial}{\partial x^a}}\left(F^{j_1\dots j_l}_{i_1\dots i_k}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}\right)\\
&= X^a\Bigg(\left(\nabla_{\frac{\partial}{\partial x^a}}F^{j_1\dots j_l}_{i_1\dots i_k}\right)\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}\\
&\qquad + \sum_b F^{j_1\dots j_l}_{i_1\dots i_k}\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_{b-1}}}\otimes{\color{red}\nabla_{\frac{\partial}{\partial x^a}}\frac{\partial}{\partial x^{j_b}}}\otimes\frac{\partial}{\partial x^{j_{b+1}}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_l}}\otimes\mathrm{d}x^{i_1}\otimes\cdots\otimes\mathrm{d}x^{i_k}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_d F^{j_1\dots j_l}_{i_1\dots i_k} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_{d-1}} \otimes \nabla_{\frac{\partial}{\partial x^a}} \mathrm{d}x^{i_d} \otimes \mathrm{d}x^{i_{d+1}} \otimes \dots \otimes \mathrm{d}x^{i_k} \Bigg) \\
&= X^a \Bigg(\left(\frac{\partial}{\partial x^a} F^{j_1\dots j_l}_{i_1\dots i_k} \right) \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k} \\
&\quad + \sum_b F^{j_1\dots j_l}_{i_1\dots i_k} \Gamma^c_{a j_b} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_{b-1}}} \otimes \frac{\partial}{\partial x^c} \otimes \frac{\partial}{\partial x^{j_{b+1}}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k} \\
&\quad + \sum_d F^{j_1\dots j_l}_{i_1\dots i_k} \Gamma^{i_d}_{ae} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_{d-1}} \otimes \mathrm{d}x^e \otimes \mathrm{d}x^{i_{d+1}} \otimes \dots \otimes \mathrm{d}x^{i_k} \Bigg) \\
&= X^a \Bigg(\left(\frac{\partial}{\partial x^a} F^{j_1\dots j_l}_{i_1\dots i_k} \right) \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k} \\
&\quad + \sum_b F^{j_1\dots j_{b-1} c j_{b+1}\dots j_l}_{i_1\dots i_k} \Gamma^{j_b}_{ac} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_{b-1}}} \otimes \frac{\partial}{\partial x^{j_b}} \otimes \frac{\partial}{\partial x^{j_{b+1}}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k} \\
&\quad - \sum_d F^{j_1\dots j_l}_{i_1\dots i_{d-1} e i_{d+1}\dots i_k} \Gamma^e_{a i_d} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_{d-1}} \otimes \mathrm{d}x^{i_d} \otimes \mathrm{d}x^{i_{d+1}} \otimes \dots \otimes \mathrm{d}x^{i_k} \Bigg) \\
&= X^a \Bigg(\frac{\partial}{\partial x^a} F^{j_1\dots j_l}_{i_1\dots i_k} + \sum_b F^{j_1\dots j_{b-1} c j_{b+1}\dots j_l}_{i_1\dots i_k} \Gamma^{j_b}_{ac} \\
&\qquad - \sum_d F^{j_1\dots j_l}_{i_1\dots i_{d-1} e i_{d+1}\dots i_k} \Gamma^e_{a i_d} \Bigg) \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k}.
\end{aligned}
$$

□

Exercise 6.4 (Hessian)**.** Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. If $u \in C^\infty(M)$ is a function on M, its **Hessian** is a $(2,0)$-tensor given by

$$\mathrm{Hess}(u)(X, Y) \coloneqq (\nabla^2 u)(X, Y),$$

where $\nabla^2 = \nabla\nabla$ and both ∇ mean total covariant derivative. Show that

$$\mathrm{Hess}(u)(X, Y) = Y(X(u)) - (\nabla_Y X)(u).$$

Solution. We show that $\nabla u = \mathrm{d}u$. For any $X \in \mathfrak{X}(M)$, we have that

$$
\begin{aligned}
(\nabla u)(X) &= \nabla_X u && \text{[definition of total covariant derivative]} \\
&= X(u) && \text{[definition of covariant derivative of a function]} \\
&= \mathrm{d}u(X) && \text{[definition of exterior derivative of a function].}
\end{aligned}
$$

We show that $\mathrm{Hess}(u)(X, Y) = Y(X(u)) - (\nabla_Y X)(u)$:

$$
\begin{aligned}
&\mathrm{Hess}(u)(X, Y) \\
&\quad = (\nabla(\nabla(u)))(X, Y) && \text{[definition of Hessian]} \\
&\quad = (\nabla_Y(\nabla u))(X) && \text{[def. of total covariant derivative]} \\
&\quad = \nabla_Y((\nabla u)(X)) - (\nabla u)(\nabla_Y X) && \text{[def. of covariant derivative of a tensor]} \\
&\quad = \nabla_Y(X(u)) - (\nabla_Y X)(u) && [\nabla u = \mathrm{d}u] \\
&\quad = Y(X(u)) - (\nabla_Y X)(u). && \text{[def. of covariant derivative of a function].}
\end{aligned}
$$

□

7 Exercise sheet No. 7 - 07-01-2021

Exercise 7.1. Let (M, g) be a Riemannian manifold with a linear connection ∇ which is compatible with g. Let $\gamma\colon I \longrightarrow M$ be a smooth embedded curve and $X, Y \in C^\infty(\gamma^*TM)$ be vector fields along γ. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{\gamma(t)}(X, Y) = g_{\gamma(t)}(\mathrm{D}_t X, Y) + g_{\gamma(t)}(X, \mathrm{D}_t Y).$$

Solution. Let $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(M)$ be extensions of $X, Y, \dot{\gamma} \in C^\infty(\gamma^*TM)$. Then,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} g_{\gamma(t)}(X, Y) \\
&\quad = \tilde{Z}_{\gamma(t)}(g(\tilde{X}, \tilde{Y})) && \text{[def. of der. of function along vector]} \\
&\quad = g_{\gamma(t)}(\nabla_{\tilde{Z}}\tilde{X}, \tilde{Y}) + g_{\gamma(t)}(\tilde{X}, \nabla_{\tilde{Z}}\tilde{Y}) && [\nabla \text{ is compatible with } g] \\
&\quad = g_{\gamma(t)}(\nabla_{\dot{\gamma}(t)}\tilde{X}, \tilde{Y}) + g_{\gamma(t)}(\tilde{X}, \nabla_{\dot{\gamma}(t)}\tilde{Y}) && [\tilde{Z} \text{ is an extension of } \dot{\gamma}] \\
&\quad = g_{\gamma(t)}(\mathrm{D}_t X, Y) + g_{\gamma(t)}(X, \mathrm{D}_t Y) && [\text{definition of } \mathrm{D}_t]. \qquad \square
\end{aligned}$$

Exercise 7.2. Let $M \subset \mathbb{R}^n$ be a submanifold of $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Equip M with the induced metric g_M from $g_{\mathbb{R}^n}$. Recall that for every $p \in M$ and $X, Y \in \mathfrak{X}(M)$ the Levi-Civita connection coming from g_M is given by

$$\nabla^M_X Y|_p = \pi_p \nabla^{\mathbb{R}^n}_{\tilde{X}} \tilde{Y}|_p,$$

where $\tilde{X}, \tilde{Y} \in \mathbb{R}^n$ are vector fields extending X, Y. Let $\gamma\colon I \longrightarrow M$ be a smooth embedded curve. The connections ∇^M and $\nabla^{\mathbb{R}^n}$ induce corresponding maps of covariant differentiation along γ, which we denote by D^M_t and $\mathrm{D}^{\mathbb{R}^n}_t$. Show that

$$\mathrm{D}^M_t X = \pi_{\gamma(t)} \mathrm{D}^{\mathbb{R}^n}_t X.$$

Solution. Let

$$\begin{aligned}
&\tilde{X} \in \mathfrak{X}(M) \text{ be an extension of } X \in C^\infty(\gamma^*TM), \\
&\tilde{\tilde{X}} \in \mathfrak{X}(\mathbb{R}^n) \text{ be an extension of } \tilde{X} \in \mathfrak{X}(M), \\
&\tilde{Z} \in \mathfrak{X}(M) \text{ be an extension of } \dot{\gamma} \in C^\infty(\gamma^*TM), \\
&\tilde{\tilde{Z}} \in \mathfrak{X}(\mathbb{R}^n) \text{ be an extension of } \tilde{Z} \in \mathfrak{X}(M).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{D}^M_t X &= \nabla^M_{\tilde{Z}} \tilde{X}|_{\gamma(t)} && [\text{definition of } \mathrm{D}^M_t] \\
&= \pi_{\gamma(t)} \nabla^{\mathbb{R}^n}_{\tilde{\tilde{Z}}} \tilde{\tilde{X}}|_{\gamma(t)} && [\nabla \text{ of a submanifold}] \\
&= \pi_{\gamma(t)} \mathrm{D}^{\mathbb{R}^n}_t X && [\text{definition of } \mathrm{D}^{\mathbb{R}^n}_t]. \qquad \square
\end{aligned}$$

Exercise 7.3 (Patallel transport is isometry)**.** Let (M, g) be a Riemannian manifold and ∇ be an affine connection on M. Show that the following are equivalent:

(a) ∇ is compatible with g.

(b) For every curve $\gamma\colon I \longrightarrow M$ and for every $b, a \in I$, the parallel transport map $P^{\gamma}_{b,a}\colon T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M$ is an isometry.

Solution. (a) $\Longrightarrow$ (b): It suffices to assume that $t_0 \in I$, $v, w \in T_{\gamma(t_0)}M$, and to prove that

$$\forall t \in I\colon g_{\gamma(t)}(P^{\gamma}_{t,t_0}v, P^{\gamma}_{t,t_0}w) = g_{\gamma(t_0)}(v, w).$$

Define a function $f\colon I \longrightarrow \mathbb{R}$ by $f(t) = g_{\gamma(t)}(P^{\gamma}_{t_0,t}v, P^{\gamma}_{t_0,t}w)$. We show that $\dot{f}(t) = 0$:

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} g_{\gamma(t)}(P^{\gamma}_{t_0,t}v, P^{\gamma}_{t_0,t}w) && \text{[by definition of } f\text{]}\\
&= g_{\gamma(t)}(\mathrm{D}_t P^{\gamma}_{t_0,t}v, P^{\gamma}_{t_0,t}w) + g_{\gamma(t)}(P^{\gamma}_{t_0,t}v, \mathrm{D}_t P^{\gamma}_{t_0,t}w) && \text{[by exercise 7.1]}\\
&= 0 && \text{[definition of parallel transport].}
\end{aligned}$$

Therefore, f is constant equal to $f(t_0) = g_{\gamma(t_0)}(v, w)$.

(b) $\Longrightarrow$ (a): It suffices to assume that $X, Y, Z \in \mathfrak{X}(M)$, that $p \in M$, and to prove that

$$X_p(g(Y, Z)) = g_p(\nabla_{X_p}Y, Z_p) + g_p(Y_p, \nabla_{X_p}Z).$$

Let $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$ be a curve such that $\gamma(0) = p$ and $\dot{\gamma}(t) = X_{\gamma(t)}$ for every $t \in (-\varepsilon, \varepsilon)$. Denote $Y_t = Y_{\gamma(t)}$ and $Z_t = Z_{\gamma(t)}$. It suffices to show that for every $t \in (-\varepsilon, \varepsilon)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g_{\gamma(t)}(Y_t, Z_t) = g_{\gamma(t)}(\mathrm{D}_t Y_t, Z_t) + g_{\gamma(t)}(Y_t, \mathrm{D}_t Z_t).$$

The proof is the following computation:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} g_{\gamma(t)}(Y_t, Z_t)\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t} g_p(P^{\gamma}_{0,t}Y_t, P^{\gamma}_{0,t}Z_t) && [P^{\gamma}_{0,t} \text{ is an isometry}]\\
&\quad= g_p\Big(\frac{\mathrm{d}}{\mathrm{d}t}P^{\gamma}_{0,t}Y_t, P^{\gamma}_{0,t}Z_t\Big) + g_p\Big(P^{\gamma}_{0,t}Y_t, \frac{\mathrm{d}}{\mathrm{d}t}P^{\gamma}_{0,t}Z_t\Big) && \text{[derivative of product]}\\
&\quad= g_{\gamma(t)}\Big(P^{\gamma}_{t,0}\frac{\mathrm{d}}{\mathrm{d}t}P^{\gamma}_{0,t}Y_t, Z_t\Big) + g_{\gamma(t)}\Big(Y_t, P^{\gamma}_{t,0}\frac{\mathrm{d}}{\mathrm{d}t}P^{\gamma}_{0,t}Z_t\Big) && [P^{\gamma}_{t,0} \text{ is an isometry}]\\
&\quad= g_{\gamma(t)}(\mathrm{D}_t Y_t, Z_t) + g_{\gamma(t)}(Y_t, \mathrm{D}_t Z_t),
\end{aligned}$$

where in the last equality we used the formula for the covariant derivative in terms of parallel transport. □

Exercise 7.4 (surface of revolution)**.** Let I be an open interval, and $a, b\colon I \longrightarrow \mathbb{R}$ be functions on I such that a is positive and the curve $\gamma(t) = (a(t), 0, b(t))$ in $\mathbb{R}^3$ is parametrized by arc-length. Define a surface of revolution $M \subset \mathbb{R}^3$ by revolving the image of γ about the z-axis.

(a) Find a local parametrization of M in a neighbourhood of any point $p \in M$ *(Hint: The coordinates of the parametrization are (θ, t), where θ is the angle for which the image of γ is rotated and t is the curve parameter of γ).*

(b) Compute the Christoffel symbols of the induced metric in the coordinates found in the first step.

(c) Show that the meridian $(\theta_0, C_1\tau + C_2)$ is a geodesic on M.

(d) Determine necessary and sufficient conditions for a latitude circle $\{t = t_0\}$ to be a geodesic.

(e) Let $c(\tau)$ be a geodesic such that $c(\tau_0) = (\theta_0, t_0)$ and $\dot{c}(\tau_0) = (\theta_1, t_1)$. Show that the following quantities constant along c:

- The Clairaut invariant: $C(\tau) = a^2(t(\tau))\dot{\theta}(\tau)$;
- The energy (which is equal to the square of the arc length): $e(\tau) = \dot{t}^2(\tau) + a^2(t(\tau))\dot{\theta}^2(\tau)$.

Solution. (a): The parametrization is $\varphi : [0, 2\pi) \times I \to M$ defined by

$$\varphi(\theta, t) = (a(t)\cos(\theta), a(t)\sin(\theta), b(t)).$$

(b): First we compute the induced metric on M. The metric on $\mathbb{R}^3$ is

$$g_{\text{st.}} = dx \otimes dx + dy \otimes dy + dz \otimes dz.$$

Let g_M denote the induced metric on M. Then we have $g_M = \varphi^* g_{\text{st.}}$. For the coordinates x, y and z we have the following relations

$$\begin{aligned} x(\theta, t) &= a(t)\cos(\theta), \\ y(\theta, t) &= a(t)\sin(\theta), \\ z(\theta, t) &= b(t). \end{aligned}$$

From here it follows that

$$\begin{aligned} dx &= \dot{a}\cos(\theta)dt - a\sin(\theta)d\theta, \\ dy &= \dot{a}\sin(\theta)dt + a\cos(\theta)d\theta, \\ dz &= \dot{b}dt. \end{aligned}$$

Then

$$\begin{aligned} dx \otimes dx &= (\dot{a}\cos(\theta)dt - a\sin(\theta)d\theta) \otimes (\dot{a}\cos(\theta)dt - a\sin(\theta)d\theta) \\ &= \dot{a}^2\cos^2(\theta)dt \otimes dt - \dot{a}a\sin(\theta)\cos(\theta)dt \otimes d\theta - a\dot{a}\sin(\theta)\cos(\theta)d\theta \otimes dt \\ &\quad + a^2\sin^2(\theta)d\theta \otimes d\theta, \end{aligned}$$

$$\begin{aligned} dy \otimes dy &= (\dot{a}\sin(\theta)dt + a\cos(\theta)d\theta) \otimes (\dot{a}\sin(\theta)dt + a\cos(\theta)d\theta) \\ &= \dot{a}^2\sin^2(\theta)dt \otimes dt + a\dot{a}\sin(\theta)\cos(\theta)dt \otimes d\theta + a\dot{a}\sin(\theta)\cos(\theta)d\theta \otimes dt \\ &\quad + a^2\cos^2(\theta)d\theta \otimes d\theta, \end{aligned}$$

and

$$dz \otimes dz = \dot{b}^2 dt \otimes dt.$$

Hence

$$g_M = \varphi^* g_{\text{st.}}$$

$$
\begin{aligned}
&= dx \otimes dx + dy \otimes dy + dz \otimes dz \\
&= \dot{a}^2 \cos^2(\theta) dt \otimes dt - \dot{a}a \sin(\theta)\cos(\theta) dt \otimes d\theta - a\dot{a}\sin(\theta)\cos(\theta) d\theta \otimes dt \\
&\quad + a^2 \sin^2(\theta) d\theta \otimes d\theta \\
&\quad + \dot{a}^2 \sin^2(\theta) dt \otimes dt + a\dot{a}\sin(\theta)\cos(\theta) dt \otimes d\theta + a\dot{a}\sin(\theta)\cos(\theta) d\theta \otimes dt \\
&\quad + a^2 \cos^2(\theta) d\theta \otimes d\theta \\
&\quad + \dot{b}^2 dt \otimes dt \\
&= (\dot{a}^2 + \dot{b}^2) dt \otimes dt + a^2 d\theta \otimes d\theta \\
&= dt \otimes dt + a^2 d\theta \otimes d\theta.
\end{aligned}
$$

So, we can write g_M as a matrix:

$$
g_M = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}.
$$

The inverse metric is given by

$$
g_M^{-1} = \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix}.
$$

Then, using the formula

$$
\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\kappa} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\kappa} \right)
$$

we compute the Christoffel symbols:

$$
\begin{aligned}
\Gamma^t_{tt} &= \frac{1}{2} g^{t\kappa} (\partial_t g_{t\kappa} + \partial_t g_{i\kappa} - \partial_\kappa g_{tt}) = \frac{1}{2} g^{tt} (\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt}) = 0 \\
\Gamma^\theta_{tt} &= \frac{1}{2} g^{t\kappa} \left(\partial_t g_{t\theta} + \partial_t g_{t\theta} - \partial_\theta g_{tt} \right) = 0 \\
\Gamma^t_{\theta t} &= \frac{1}{2} g^{t\kappa} \left(\partial_\theta g_{t\kappa} + \partial_t g_{\theta\kappa} - \partial_\kappa g_{\theta t} \right) = 0 \\
\Gamma^t_{t\theta} &= 0 \\
\Gamma^\theta_{\theta t} &= \frac{1}{2} g^{\theta\kappa} \left(\partial_\theta g_{t\kappa} + \partial_t g_{\theta\kappa} - \partial_\kappa g_{\theta t} \right) = \frac{1}{2} g^{\theta\theta} \partial_t g_{\theta\theta} = \frac{\dot{a}}{a} \\
\Gamma^\theta_{t\theta} &= \frac{\dot{a}}{a} \\
\Gamma^t_{\theta\theta} &= \frac{1}{2} g^{t\kappa} \left(\partial_\theta g_{\theta\kappa} + \partial_\theta g_{\theta\kappa} - \partial_\kappa g_{\theta\theta} \right) = -\frac{1}{2} g^{tt} \partial_t g_{\theta\theta} = -a\dot{a} \\
\Gamma^\theta_{\theta\theta} &= 0.
\end{aligned}
$$

(c): Let $c : J \to M$ be a curve in M and denote by τ the time parameter of c. In the coordinates given by φ the curve c may be written in the form $c : J \to [0, 2\pi) \times I$ with $c(\tau) = (\theta(\tau), t(\tau))$. The geodesic equation, in these coordinates in the of the form

$$
\begin{aligned}
\ddot{\theta} + \dot{\theta}^2 \Gamma^\theta_{\theta\theta} + 2\dot{\theta}\dot{t}\Gamma^\theta_{\theta t} + \dot{t}^2 \Gamma^\theta_{tt} &= 0, \\
\ddot{t} + \dot{\theta}^2 \Gamma^t_{\theta\theta} + 2\dot{\theta}\dot{t}\Gamma^t_{\theta t} + \dot{t}^2 \Gamma^t_{tt} &= 0.
\end{aligned}
$$

By (b) these equations reduce to

$$
\ddot{\theta} + 2\dot{\theta}\dot{t}\frac{\dot{a}}{a} = 0,
$$

$$\ddot{t} - \dot{\theta}^2 a\dot{a} = 0.$$

For meridians the curve c is of the form $c(\tau) = (\theta_0, t(\tau))$. From the geodesic equations, we see that only the second one survives and we obtain

$$\ddot{t} = 0.$$

Hence $t(\tau) = C_1\tau + C_2$ where $C_1, C_2 \in \mathbb{R}$.

(d): Consider a longitudinal curve, which we parametrize as $c(\tau) = (\theta(\tau), t_0)$. Then the geodesic equation from part (3) go over in

$$\ddot{\theta}(\tau) = 0, \quad \text{and} \quad \dot{\theta}^2 a\dot{a} = 0.$$

The first equation implies that θ is of the form $\theta(\tau) = C_1\tau + C_2$. This inserted in the second equation implies that

$$C_1^2 a\dot{a} = 0.$$

Since it is assumed that $a(t_0) > 0$ then $C_1^2\dot{a}(t_0) = 0$. Which implies that $C_1 = 0$ or $\dot{a}(t_0) = 0$. If $C_1 = 0$ then c is just a point. If there exists a $t_0 \in I$ such that $\dot{a}(t_0) = 0$ then $(C_1\tau + C_2, t_0)$ is a longitudinal geodesic.

(e): We prove that the Clairaut invariant is constant along c.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}C(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}a^2(t(\tau))\dot{\theta}(\tau)\\ &= 2a\dot{a}\dot{t}\dot{\theta} + a^2\ddot{\theta}\\ &= a(a\ddot{\theta} + 2\dot{\theta}\dot{t}\dot{a})\\ &= 0.\end{aligned}$$

We prove that the energy $\dot{t}^2(\tau) + a^2(t(\tau))\dot{\theta}^2(\tau)$ is constant along c. Differentiating with respect to τ yields

$$\begin{aligned}\frac{d}{d\tau}e(\tau) &= \frac{d}{d\tau}\dot{t}^2(\tau) + a^2(t(\tau))\dot{\theta}^2(\tau)\\ &= 2\dot{t}\ddot{t} + 2a\dot{a}\dot{t}\dot{\theta}^2 + 2a^2\dot{\theta}\ddot{\theta}\\ &= 2\dot{t}\ddot{t} + 4a\dot{a}\dot{t}\dot{\theta}^2 - 2a\dot{a}\dot{t}\dot{\theta}^2 + 2a^2\dot{\theta}\ddot{\theta}\\ &= 2\dot{t}(\ddot{t} - a\dot{a}\dot{\theta}^2) + 2a\dot{\theta}(a\ddot{\theta} + 2\dot{a}\dot{t}\dot{\theta})\\ &= 0.\end{aligned}$$

□

Exercise 7.5. Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$ be such that $\|(\nabla f)(p)\|_{g(p)} = 1$ for all $p \in M$. Show that the integral curves of ∇f are geodesics.

Solution. Let γ be an integral curve of ∇f, i.e. $\dot{\gamma}(t) = \nabla f(\gamma(t))$. We want to show that $\mathrm{D}_t\dot{\gamma} = 0$. For this, it suffices to show that $\nabla_{\nabla f}\nabla f = 0$, since $\mathrm{D}_t\dot{\gamma} = \nabla_{\nabla f}\nabla f|_{\gamma(t)}$.

We show that $g(\nabla_X\nabla f, Y) = g(\nabla_Y\nabla f, X)$, for all $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned}0 &= d^2 f(X, Y)\\ &= X(df(Y)) - Y(df(X)) - df([X, Y])\\ &= X(g(\nabla f, Y)) - Y(g(\nabla f, X)) - g(\nabla f, [X, Y])\end{aligned}$$

$$
\begin{aligned}
&= g(\nabla_X \nabla f, Y) + g(\nabla f, \nabla_X Y) - g(\nabla_Y \nabla f, X) - g(\nabla f, \nabla_Y X) - g(\nabla f, [X, Y]) \\
&= g(\nabla_X \nabla f, Y) - g(\nabla_Y \nabla f, X) + g(\nabla f, \nabla_X Y - \nabla_Y X - [X, Y]) \\
&= g(\nabla_X \nabla f, Y) - g(\nabla_Y \nabla f, X).
\end{aligned}
$$

We show that $g(\nabla_X \nabla f, \nabla f) = 0$, for all $X \in \mathfrak{X}(M)$:

$$
\begin{aligned}
0 &= X(1) \\
&= X(g(\nabla f, \nabla f)) \\
&= g(\nabla_X \nabla f, \nabla f) + g(\nabla f, \nabla_X \nabla f) \\
&= 2g(\nabla_X \nabla f, \nabla f).
\end{aligned}
$$

Then, for all $X \in \mathfrak{X}(M)$:

$$
\begin{aligned}
g(\nabla_{\nabla f} \nabla f, X) &= g(\nabla_X \nabla f, \nabla f) \\
&= 0.
\end{aligned}
$$

Therefore $\nabla_{\nabla f} \nabla f = 0$. □

Exercise 7.6 (homogeneous Riemannian manifold)**.** Let (M, g) be a Riemannian manifold. Consider the group of isometries of M:

$$
\mathrm{Isom}(M, g) = \{\phi \colon M \longrightarrow M \mid \phi \text{ is an isometry}\}.
$$

This group acts on M via

$$
\begin{aligned}
\mathrm{Isom}(M, g) \times M &\longrightarrow M \\
(\phi, p) &\longmapsto \phi(p).
\end{aligned}
$$

M is **homogeneous** if this action is transitive (i.e. for all $p, q \in M$ there exists a $\phi \in \mathrm{Isom}(M, g)$ such that $\phi(p) = q$). Show that if M is homogeneous then M is geodesically complete.

Solution. It suffices to assume that $p \in M$, $v \in T_pM$ has unit norm and that $\gamma \colon I \longrightarrow M$ is a geodesic with maximal interval of definition I such that $0 \in I$, $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, and to prove that $I = \mathbb{R}$. Assume by contradiction that $I = (a, b)$, with $a \neq -\infty$ or $b \neq +\infty$.

We derive a contradiction in the case where $b \neq +\infty$. There exists $r > 0$ such that $\exp_p$ is defined on $B_r(0) \subset T_pM$. Choose $\varepsilon \in (0, r)$ and define $q = \gamma(b - \varepsilon)$ and $w = \dot{\gamma}(b - \varepsilon)$. Since M is homogeneous, there exists an isometry $\phi \colon M \longrightarrow M$ such that $\phi(p) = q$. Since ϕ is an isometry, $\exp_q$ is defined on $B_r(0) \subset T_qM$. Define a curve $\overline{\gamma} \colon (a, b - \varepsilon + r) \longrightarrow M$ by the equation

$$
\overline{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } t \in (a, b) \\ \exp_q((t - b + \varepsilon)w) & \text{if } t \in (b - \varepsilon - r, b - \varepsilon + r) \end{cases}.
$$

Notice that $\exp_q((t-b+\varepsilon)w)$ is well defined if $t \in (b-\varepsilon-r, b-\varepsilon+r)$ because $\exp_q$ is well defined on $B_r(0) \subset T_qM$ and w has unit norm. We need to check that $\overline{\gamma}$ is well defined, i.e. that if $t \in (a,b) \cap (b-\varepsilon-r, b-\varepsilon+r) = (b-\varepsilon-r, b)$ then $\gamma(t) = \exp_q((t-b+\varepsilon)w)$. This is true, by the following informal computation:

$$\begin{aligned}
\exp_q((t-b-\varepsilon)w) &= \exp_{\exp_p((b-\varepsilon)v)}((t-b-\varepsilon)w) \\
&= \text{start at } p, \text{ flow in the direction of } v \text{ for } b-\varepsilon \text{ seconds} \\
&\quad\ \text{(so now we are at } q \text{ with tangent vector } w\text{)} \\
&\quad\ \text{then flow again for } t-b-\varepsilon \text{ seconds in the direction of } w \\
&= \text{start at } p, \text{ flow in the direction of } v \text{ for } t \text{ seconds} \\
&= \exp_p(tv) \\
&= \gamma(t).
\end{aligned}$$

Then, $\overline{\gamma}$ is a geodesic and it is an extension of γ from (a,b) to $(a, b-\varepsilon+r)$. This contradicts the fact that I was the maximal interval of definition.

To derive a contradiction in the case where $a \neq -\infty$, proceed analogously as in the proof of the case $b \neq +\infty$. $\square$

8 Exercise sheet No. 8 - 14-01-2021

Exercise 8.1 (naturality of exponential map)**.** Let $\phi : (M, g_M) \to (N, g_N)$ be an isometry. Denote by $\mathcal{O}_p^M \subset T_pM$ the domain of $\exp_p^M$ and by $\mathcal{O}_{\phi(p)}^N \subset T_{\phi(p)}N$ the domain of $\exp_{\phi(p)}^N$. Show that $\mathrm{D}\phi(p)(\mathcal{O}_p^M) = \mathcal{O}_{\phi(p)}^N$ and that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_p^M \arrow[r, "\mathrm{D}\phi(p)"] \arrow[d, "\exp_p^M"'] & \mathcal{O}_{\phi(p)}^N \arrow[d, "\exp_{\phi(p)}^N"] \\
M \arrow[r, "\phi"'] & N
\end{tikzcd}.$$

Solution. We show that $\mathrm{D}\phi|_p(\mathcal{O}_p^M) = \mathcal{O}_{\phi(p)}^N$. To show that $\mathrm{D}\phi|_p(\mathcal{O}_p^M) \subset \mathcal{O}_{\phi(p)}^N$ it suffices to assume that $v \in \mathcal{O}_p^M \subset T_pM$ (i.e. that the geodesic γ^M starting at $p \in M$ with initial velocity $v \in T_pM$ exists for $t \in [0,1]$) and to prove that $\mathrm{D}\phi|_p v \in \mathcal{O}_{\phi(p)}^N$ (i.e. that the geodesic γ^N starting at $\phi(p)$ with initial velocity $\mathrm{D}\phi|_p v$ exists for $t \in [0,1]$). By naturality of the Levi-Civita connection, $\phi \circ \gamma^M$ is a geodesic. Since

$$\gamma^N(0) = \phi(p) = \phi \circ \gamma^M(0)$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma^N(t) = \mathrm{D}\phi|_p v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi \circ \gamma^M(t)$$

and by uniqueness of geodesics with prescribed initial conditions, we conclude that $\gamma^N = \phi \circ \gamma^M$. This proves that $\gamma^N(t)$ is defined for $t \in [0,1]$. To show that $\mathrm{D}\phi|_p(\mathcal{O}_p^M) \supset \mathcal{O}_{\phi(p)}^N$, we apply the previous inclusion but reversing the roles of M and N and considering the isometry $\phi^{-1} \colon N \longrightarrow M$.

We show that the diagram commutes. For this, it suffices to assume that $v \in \mathcal{O}_p^M \subset T_pM$ and to prove that $\phi(\exp_p^M(v)) = \exp_{\phi(p)}^N(\mathrm{D}\phi|_p v)$. Denote by γ^M the geodesic in M which starts at $p \in M$ with initial velocity v and denote by γ^N the geodesic in N which starts at $\phi(p) \in N$ with initial velocity $\mathrm{D}\phi|_p v$. By the same argument as in the first part of this proof, $\gamma^N = \phi \circ \gamma^M$. Then,

$$\begin{aligned}
\phi(\exp_p^M(v)) &= \phi \circ \gamma^M(1) && [\text{definition of } \exp^M] \\
&= \gamma^N(1) && [\gamma^N = \phi \circ \gamma^M] \\
&= \exp_{\phi(p)}^N(\mathrm{D}\phi|_p v) && [\text{definition of } \exp^N]. \qquad \square
\end{aligned}$$

Exercise 8.2 (local isometries)**.** Let (M, g_M), (N, g_N) be Riemannian manifolds. A smooth map $\phi \colon M \longrightarrow N$ is a **local isometry** if for every $p \in M$ there exists U a neighbourhood of p in M and V a neighbourhood of $\phi(p)$ in N such that $\phi(U) = V$ and $\phi \colon U \longrightarrow V$ is an isometry. Assume that M is connected and that $\phi, \psi \colon M \longrightarrow N$ are local isometries such that there exists $p \in M$ with $\phi(p) = \psi(p)$ and $\mathrm{D}\phi|_p = \mathrm{D}\psi|_p \colon T_pM \longrightarrow T_{\phi(p)}N$. Show that $\phi = \psi$.

Solution. Define

$$S = \{p \in M \mid \phi(p) = \psi(p) \text{ and } \mathrm{D}\phi(p) = \mathrm{D}\psi(p)\}.$$

We want to show that $S = M$. S is closed and by assumption there exists $p \in S$. Then, since M is connected it suffices to show that S is open. For this, it suffices to assume that

$p \in S \subset M$ and to prove that there exists a $V \subset M$ open such that $p \in V \subset S \subset M$. Let $q = \phi(p) = \psi(p)$ and $T = \mathrm{D}\phi(p) = \mathrm{D}\psi(q)$. Choose $U, U_\phi, U_\psi, V, V_\phi, V_\psi$ open such that in the following diagram every arrow is well defined (i.e. maps its domain to its target) and a bijection and the following diagram commutes:

$$\begin{tikzcd}
T_{\psi(p)}N & T_pM & T_{\phi(p)}N \\
\cup & \cup & \cup \\
U_\psi \arrow[d, "\exp^N_{\psi(p)}"'] & U \arrow[l, "\mathrm{D}\psi(p)"'] \arrow[r, "\mathrm{D}\phi(p)"] \arrow[d, "\exp^M_p"] & U_\phi \arrow[d, "\exp^N_{\phi(p)}"] \\
V_\psi & V \arrow[l, "\psi"] \arrow[r, "\phi"'] & V_\phi \\
\cap & \cap & \cap \\
N & M & N
\end{tikzcd}.$$

This can be achieved by using the fact that ϕ, ψ are local isometries, by exercise 8.1, and by rescaling the open sets appropriately. By construction, $p \in V$. We show that $V \subset S$. For every $u \in U$,

$$\begin{aligned}
\psi(\exp^M_p(u)) &= \exp^N_{\psi(p)}(\mathrm{D}\psi(p)u) \\
&= \exp^N_q(Tu) \\
&= \exp^N_{\phi(p)}(\mathrm{D}\phi(p)u) \\
&= \phi(\exp^M_p(u)).
\end{aligned}$$

Since $\exp^M_p \colon U \longrightarrow V$ is a bijection, we conclude that $\phi = \psi$ on the open set V. This implies that $\mathrm{D}\phi(p) = \mathrm{D}\psi(p)$ for every $p \in V$. $\square$

Exercise 8.3 (complete submanifold)**.**

(a) Let N be a complete Riemannian manifold and M be an embedded submanifold of N such that M is a closed subset of N. Show that M is complete.

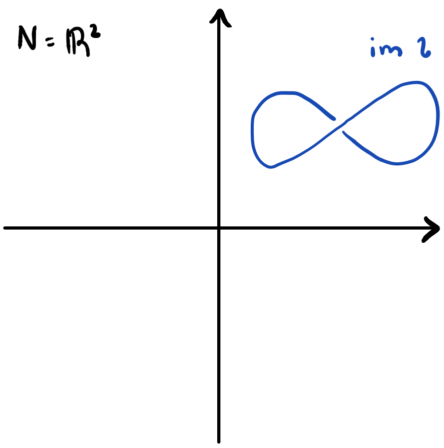
(b) Give an example of a complete Riemannian manifold N and a closed immersed submanifold M of N such that M is not complete.

Solution. (a): It suffices to assume that $(x_n)_n \subset M$ is a sequence in M which is Cauchy with respect to d^M and to prove that $(x_n)_n$ converges to $x \in M$ with respect to d^M. Notice that $d^M(x,y) \geq d^N(x,y)$, because there are more curves in N than in M joining x and y. Also, since M is an embedded submanifold, the topology of M is equal to the subspace topology.

$$\begin{aligned}
&(x_n)_n \subset M \text{ is Cauchy with respect to } d^M \\
&\quad\Longrightarrow (x_n)_n \subset M \text{ is Cauchy with respect to } d^N && [d^M(x,y) \geq d^N(x,y)] \\
&\quad\Longrightarrow (x_n)_n \subset N \text{ converges to } x \in N \text{ with respect to } d^N && [N \text{ is complete}] \\
&\quad\Longrightarrow (x_n)_n \subset M \text{ converges to } x \in M \text{ with respect to } d^M && [M \text{ is closed, embedded}].
\end{aligned}$$

(b): Consider $N = \mathbb{R}^2$ and $M = (0, +\infty)$, with immersion $\iota \colon M \longrightarrow N$ represented in figure 8.1:

Then, N is complete, $\iota(M) \subset N$ is closed, and ι is an immersion, but M is not complete. $\square$



Exercise 8.4 (adjoint representations)**.** Let G be a Lie group.

(a) Define $C\colon G \longrightarrow \mathrm{Diff}(G)$, $g \longmapsto C_g$, via $C_g(h) = ghg^{-1}$. Show that C is a Lie group action (the group action by **conjugation**), i.e. that $C_{gh} = C_g C_h$.

(b) Define $\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g})$, $g \longmapsto \mathrm{Ad}_g$, via $\mathrm{Ad}_g(X) = \mathrm{D}C_g(e)X$. Show that Ad is a Lie group representation (the **adjoint representation** of G), i.e. that $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h$.

(c) Define $\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, $X \longmapsto \mathrm{ad}_X$, by $\mathrm{ad}_X(Y) = [X, Y]$. Show that ad is a Lie algebra representation (the **adjoint representation** of $\mathfrak{g}$), i.e. that $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

(d) Show that $\mathrm{ad}_X = \mathrm{D}\,\mathrm{Ad}(e)X$.

(e) Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ and g be the unique left-invariant Riemannian metric on G such that $g_e = \langle\cdot,\cdot\rangle$. Show that g is right invariant if and only if $\langle\cdot,\cdot\rangle$ is Ad-invariant.

(f) Show that if $\langle\cdot,\cdot\rangle$ is Ad-invariant then ad_X is anti-symmetric with respect to $\langle\cdot,\cdot\rangle$, i.e. $\langle \mathrm{ad}_X Y, Z\rangle + \langle Y, \mathrm{ad}_X Z\rangle = 0$.

Solution. (a):

$$\begin{aligned}
C_{gh}(l) &= ghl(gh)^{-1} && \text{[definition of } C\text{]}\\
&= ghlh^{-1}g^{-1} \\
&= gC_h(l)g^{-1} && \text{[definition of } C\text{]}\\
&= C_gC_h(l) && \text{[definition of } C\text{].}
\end{aligned}$$

(b):

$$\begin{aligned}
\mathrm{Ad}_{gh} &= \mathrm{D}C_{gh}(e) && \text{[definition of Ad]}\\
&= \mathrm{D}C_g(e)\mathrm{D}C_h(e) && \text{[chain rule]}\\
&= \mathrm{Ad}_g \mathrm{Ad}_h && \text{[definition of Ad].}
\end{aligned}$$

(c):

$$\begin{aligned}
\operatorname{ad}_{[X,Y]} Z &= [[X,Y],Z] && \text{[definition of ad]}\\
&= -[[Y,Z],X] - [[Z,X],Y] && \text{[Jacobi identity]}\\
&= [X,[Y,Z]] - [Y,[X,Z]] && \text{[$[\cdot,\cdot]$ is anti-symmetric]}\\
&= \operatorname{ad}_X([Y,Z]) - \operatorname{ad}_Y([X,Z]) && \text{[definition of ad]}\\
&= \operatorname{ad}_X \operatorname{ad}_Y(Z) - \operatorname{ad}_Y \operatorname{ad}_X(Z) && \text{[definition of ad]}\\
&= [\operatorname{ad}_X, \operatorname{ad}_Y](Z) && \text{[definition of $[\cdot,\cdot]$ on $\mathfrak{gl}(\mathfrak{g})$].}
\end{aligned}$$

(d): It suffices to assume that $V, W \in \mathfrak{g}$ and to prove that $(\mathrm{D}\operatorname{Ad}(e)V)W = \operatorname{ad}_V W$. Denote by X^V, X^W the left invariant vector fields in G that equal V, W at the identity.

$$\begin{aligned}
&(\mathrm{D}\operatorname{Ad}(e)V)W\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \operatorname{Ad}_{\exp(tV)} W && \text{[definition of derivative]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{D}C_{\exp(tV)}(e)W && \text{[definition of Ad]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} C_{\exp(tV)}(\exp(sW)) && \text{[definition of derivative]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \exp(tV)\exp(sW)\exp(-tV) && \text{[definition of C]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \phi^t_{X^V}(e)\cdot\phi^s_{X^W}(e)\cdot\phi^{-t}_{X^V}(e) && \text{[definition of exp]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \phi^t_{X^V}(e)\cdot\phi^{-t}_{X^V}(\phi^s_{X^W}(e)) && \text{[X^V is left invariant]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \phi^{-t}_{X^V}(e)(\phi^t_{X^V}(e)\cdot\phi^s_{X^W}(e)) && \text{[X^V is left invariant]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \phi^{-t}_{X^V}(\phi^s_{X^W}(\phi^t_{X^V}(e))) && \text{[X^W is left invariant]}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \phi^{-t}_{X^V}\circ\phi^s_{X^W}(\phi^t_{X^V}(e))\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} ((\phi^{-t}_{X^V})_* X^W)|_e && \text{[definition of push forward]}\\
&= L_{X^V} X^W|_e && \text{[definition of Lie derivative]}\\
&= [X^V, X^W]^G_e && \text{[$L_X Y = [X,Y]$]}\\
&= [V,W]^{\mathfrak{g}} && \text{[definition of $[\cdot,\cdot]^{\mathfrak{g}}$]}\\
&= \operatorname{ad}_V W && \text{[definition of ad].}
\end{aligned}$$

(e): We show that g is right-invariant if and only if $g_e = \langle\cdot,\cdot\rangle$ is Ad-invariant:

$$\begin{aligned}
&g \text{ is right-invariant}\\
&\quad\iff \forall h \in G\colon R_h^* g = g\\
&\quad\iff \forall h, l \in G\colon \forall u, v \in T_l G\colon\\
&\qquad\quad g_{lh}\Big(\mathrm{D}R_h(l)\cdot u, \mathrm{D}R_h(l)\cdot v\Big) = g_l\Big(u, v\Big)\\
&\quad\iff \forall h, l \in G\colon \forall u, v \in T_l G\colon
\end{aligned}$$

$$
\begin{aligned}
&g_e\Big(\mathrm{D}L_{(lh)^{-1}}(lh)\mathrm{D}R_h(l)\cdot u, \mathrm{D}L_{(lh)^{-1}}(lh)\mathrm{D}R_h(l)\cdot v\Big)\\
&\quad = g_e\Big(\mathrm{D}L_{l^{-1}}(l)\cdot u, \mathrm{D}L_{l^{-1}}(l)\cdot v\Big)\\
\iff &\forall h,l\in G\colon \forall u,v\in T_lG\colon\\
&g_e\Big(\mathrm{D}L_{h^{-1}}(h)\mathrm{D}R_h(e)\mathrm{D}L_{l^{-1}}(l)\cdot u, \mathrm{D}L_{h^{-1}}(h)\mathrm{D}R_h(l)\mathrm{D}L_{l^{-1}}(l)\cdot v\Big)\\
&\quad = g_e\Big(\mathrm{D}L_{l^{-1}}(l)\cdot u, \mathrm{D}L_{l^{-1}}(l)\cdot v\Big)\\
\iff &\forall h\in G\colon \forall u,v\in T_eG\colon\\
&g_e\Big(\mathrm{D}L_h(h^{-1})\mathrm{D}R_{h^{-1}}(e)\cdot u, \mathrm{D}L_h(h^{-1})\mathrm{D}R_{h^{-1}}(e)\cdot v\Big) = g_e\Big(u,v\Big)\\
\iff &\forall h\in G\colon \forall u,v\in T_eG\colon g_e(\mathrm{Ad}_h u, \mathrm{Ad}_h v) = g_e(u,v)\\
\iff &g_e \text{ is right invariant.}
\end{aligned}
$$

(f): For all $t\in\mathbb{R}$, $\langle Y,Z\rangle = \langle \mathrm{Ad}_{\exp(tX)} Y, \mathrm{Ad}_{\exp(tX)} Z\rangle$. Differentiating this expression,

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle Y,Z\rangle\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle \mathrm{Ad}_{\exp(tX)} Y, \mathrm{Ad}_{\exp(tX)} Z\rangle\\
&= \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Ad}_{\exp(tX)} Y, \mathrm{Ad}_{\exp(tX)} Z\Big\rangle + \Big\langle \mathrm{Ad}_{\exp(tX)} Y, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Ad}_{\exp(tX)} Z\Big\rangle\\
&= \langle \mathrm{ad}_X Y, Z\rangle + \langle Y, \mathrm{ad}_X Z\rangle.
\end{aligned}
$$ □

Exercise 8.5 (bi-invariant metric)**.** Let G be a compact Lie group. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g} = T_eG$ and $\mathrm{d}h$ be a Right invariant measure on G. Define g to be the unique left-invariant Riemannian metric on G which at the identity is given by

$$
g_e(u,v) = \int_G \langle \mathrm{Ad}_h u, \mathrm{Ad}_h v\rangle \mathrm{d}h
$$

for all $u,v\in\mathfrak{g}=T_eG$. Show that g is right-invariant.

Solution. We show that g_e is Ad-invariant. For this, it suffices to assume that $l\in G$, $u,v\in\mathfrak{g}$ and to prove that $g_e(\mathrm{Ad}_l u, \mathrm{Ad}_l v) = g_e(u,v)$.

$$
\begin{aligned}
&g_e(\mathrm{Ad}_l u, \mathrm{Ad}_l v)\\
&\quad= \int_G \langle \mathrm{Ad}_h \mathrm{Ad}_l u, \mathrm{Ad}_h \mathrm{Ad}_l v\rangle \mathrm{d}h && [\text{definition of } g_e]\\
&\quad= \int_G \langle \mathrm{Ad}_{hl} u, \mathrm{Ad}_{hl} v\rangle \mathrm{d}h && [\text{Ad is a representation}]\\
&\quad= \int_G \langle \mathrm{Ad}_{hl} u, \mathrm{Ad}_{hl} v\rangle \mathrm{d}(hl) && [\mathrm{d}h \text{ is right invariant}]\\
&\quad= \int_G \langle \mathrm{Ad}_{h} u, \mathrm{Ad}_{h} v\rangle \mathrm{d}(h) && [\text{change of variables}]\\
&\quad= g_e(u,v) && [\text{definition of } g_e].
\end{aligned}
$$ □

Exercise 8.6 (Riemannian and Lie group exponential)**.** Let G be a Lie group, with a bi-invariant Riemannian metric g. Denote by $\exp_{RM}\colon T_eG\longrightarrow G$ the Riemannian exponential map and denote by $\exp_{LG}\colon T_eG\longrightarrow G$ the Lie group exponential map.

(a) Show that for all $X,Y\in\mathfrak{X}(G)$ left invariant vector fields on G we have that $\nabla_X Y = \frac{1}{2}[X,Y]$.

(b) Conclude that if X is a left invariant vector field, then the flow lines of X are geodesics and that $\exp_{RM} = \exp_{LG}$.

Solution. (a): It suffices to assume that X, Y, Z are left invariant vector fields and to prove that $2(\nabla_X Y, Z) = \langle [X, Y], Z\rangle$.

$$\begin{aligned}
&2\langle \nabla_X Y, Z\rangle \\
&\quad = X \cdot \langle Y, Z\rangle + Y \cdot \langle X, Z\rangle - Z \cdot \langle X, Y\rangle \\
&\qquad - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle && \text{[Koszul formula]} \\
&\quad = \ - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle && [\langle X, Y\rangle, \langle X, Z\rangle, \langle Y, Z\rangle \text{ are constant}] \\
&\quad = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle && \text{[reorder terms]} \\
&\quad = \langle [X, Y], Z\rangle && \text{[by exercise 8.4].}
\end{aligned}$$

(b): If X is a left invariant vector field, then it's flow lines are geodesics because $\nabla_X X = 1/2[X, X] = 0$. To show that $\exp_{RM} = \exp_{LG}$, it suffices to assume that X is a left invariant vector field in G and to prove that $\exp_{RM}(X_e) = \exp_{LG}(X_e)$. By definition of $\exp_{RM}$, $\exp_{RM}(X_e) = \gamma(1)$ where γ is the unique geodesic such that $\gamma(0) = e$ and $\dot\gamma(0) = X_e$. By definition of $\exp_{LG}$, $\exp_{LG}(X_e) = \phi_X^1(e)$, where ϕ_X^t is the time-t flow of X. Since flow lines of X are geodesics, $\phi_X^1(e) = \gamma(1)$. $\square$

9 Exercise sheet No. 9 - 21-01-2021

Exercise 9.1 (curvature of Lie group, from [GN14]). Let G be a Lie group with a bi-invariant metric g. Show that if X, Y, Z are left invariant vector fields on G, then

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]].$$

Solution. Recall that if X, Y are left invariant vector fields then

$$\nabla_X Y = \frac{1}{2}[X,Y]. \tag{1}$$

Then,

$$\begin{aligned}
R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z && \text{[by definition of curvature]}\\
&= \frac{1}{2}\nabla_X[Y,Z] - \frac{1}{2}\nabla_Y[X,Z] - \nabla_{[X,Y]}Z && \text{[by (1)]}\\
&= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] && \text{[by (1)]}\\
&= \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[Z,X]] + \frac{1}{2}[Z,[X,Y]] && \text{[$[\cdot,\cdot]$ is antisymmetric]}\\
&= \frac{1}{4}[Z,[X,Y]] && \text{[Jacobi identity].}
\end{aligned}$$

□

Exercise 9.2 (rescaling the metric, from [GN14]). Let M be a manifold, $\rho > 0$ be a real number, and g_1, g_2 be Riemannian metrics on M such that $g_1 = \rho g_2$. Show that

(a) $\nabla^1_X Y = \nabla^2_X Y$;

(b) $R_1(X,Y)Z = R_2(X,Y)Z$ and $R_1(X,Y,Z,W) = \rho R_2(X,Y,Z,W)$;

(c) $\kappa_1(X,Y) = \rho^{-1}\kappa_2(X,Y)$, for $X, Y \in T_pM$ linearly independent;

(d) $\mathrm{Ric}_1 = \mathrm{Ric}_2$;

(e) $S_1 = \rho^{-1}S_2$.

Solution. (a):

$$\begin{aligned}
2\langle\nabla^1_X Y, Z\rangle_1 &= X\langle Y,Z\rangle_1 + Y\langle X,Z\rangle_1 - Z\langle X,Y\rangle_1 && \text{[Koszul formula for ∇^1]}\\
&\quad -\langle[X,Z],Y\rangle_1 - \langle[Y,Z],X\rangle_1 - \langle[X,Y],Z\rangle_1 &&\\
&= \rho X\langle Y,Z\rangle_2 + \rho Y\langle X,Z\rangle_2 - \rho Z\langle X,Y\rangle_2 && \text{[$g_1 = \rho g_2$]}\\
&\quad -\rho\langle[X,Z],Y\rangle_2 - \rho\langle[Y,Z],X\rangle_2 - \rho\langle[X,Y],Z\rangle_2 &&\\
&= 2\rho\langle\nabla^2_X Y, Z\rangle_2 && \text{[Koszul formula for ∇^2]}\\
&= 2\langle\nabla^2_X Y, Z\rangle_1 && \text{[$g_1 = \rho g_2$].}
\end{aligned}$$

(b):

$$\begin{aligned}
R_1(X,Y)Z &= \nabla^1_X\nabla^1_Y Z - \nabla^1_Y\nabla^1_X Z - \nabla^1_{[X,Y]}Z && \text{[definition of R_1 as a $(3,1)$-tensor]}\\
&= \nabla^2_X\nabla^2_Y Z - \nabla^2_Y\nabla^2_X Z - \nabla^2_{[X,Y]}Z && \text{[by (a)]}
\end{aligned}$$

$$
\begin{aligned}
&= R_2(X,Y)Z && \text{[definition of } R_2 \text{ as a } (3,1)\text{-tensor]}.
\end{aligned}
$$

and

$$
\begin{aligned}
R_1(X,Y,Z,W) &= \langle R_1(X,Y)Z, W\rangle_1 && \text{[definition of } R_1 \text{ as a } (4,0)\text{-tensor]} \\
&= \langle R_2(X,Y)Z, W\rangle_1 && \text{[by the computation above]} \\
&= \rho\langle R_2(X,Y)Z, W\rangle_2 && [g_1 = \rho g_2] \\
&= \rho R_2(X,Y,Z,W) && \text{[definition of } R_2 \text{ as a } (4,0)\text{-tensor]}.
\end{aligned}
$$

(c):

$$
\begin{aligned}
\kappa_1(X,Y) &= \frac{R_1(X,Y,Y,X)}{\|X\|_1^2\|Y\|_1^2 - \|X,Y\|_1} && \text{[definition of } \kappa_1] \\
&= \frac{\rho R_2(X,Y,Y,X)}{\rho^2\|X\|_2^2\|Y\|_2^2 - \rho^2\|X,Y\|_2} && \text{[by (b)]} \\
&= \frac{1}{\rho}\kappa_2(X,Y) && \text{[by definition of } \kappa_2].
\end{aligned}
$$

(d): It suffices to assume that $p \in M$, $X, Y \in T_pM$, and to prove that $(\mathrm{Ric}_1)_p(X,Y) = (\mathrm{Ric}_2)_p(X,Y)$. For $i = 1, 2$, define a linear map $T_i \colon T_pM \longrightarrow T_pM$ via $T_i(Z) = R_i(X,Y)Z$. Then, $T_1 = T_2$:

$$
\begin{aligned}
T_1(Z) &= R_1(X,Y)Z && \text{[by definition of } T_1] \\
&= R_2(X,Y)Z && \text{[by (b)]} \\
&= T_2(Z) && \text{[by definition of } T_2].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\mathrm{Ric}_1)_p(X,Y) &= \operatorname{tr} T_1 && \text{[by definition of } \mathrm{Ric}_1] \\
&= \operatorname{tr} T_2 && \text{[by the above computation]} \\
&= (\mathrm{Ric}_2)_p(X,Y) && \text{[by definition of } \mathrm{Ric}_2].
\end{aligned}
$$

(e): It suffices to assume that $p \in M$ and to prove that $S_1(p) = S_2(p)$. Choose $E_1, \ldots, E_n$ a g_1-orthonormal basis of T_pM. Then, $\sqrt{\rho}E_1, \ldots, \sqrt{\rho}E_n$ is a g_2-orthonormal basis of T_pM:

$$
\begin{aligned}
\langle \sqrt{\rho}E_i, \sqrt{\rho}E_j\rangle_2 &= \frac{1}{\rho}\langle \sqrt{\rho}E_i, \sqrt{\rho}E_j\rangle_1 \\
&= \langle E_i, E_j\rangle_1 \\
&= \delta_{ij}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_1(p) &= \sum_{i=1}^n \mathrm{Ric}_1(E_i, E_i) && \text{[by definition of } S_1] \\
&= \sum_{i=1}^n \mathrm{Ric}_2(E_i, E_i) && \text{[by (d)]} \\
&= \frac{1}{\rho}\sum_{i=1}^n \mathrm{Ric}_2(\sqrt{\rho}E_i, \sqrt{\rho}E_i) && [\mathrm{Ric}_2 \text{ is a tensor, hence bilinear}] \\
&= \frac{1}{\rho}S_2(p) && \text{[by definition of } S_2]. \qquad \square
\end{aligned}
$$

Exercise 9.3 (from [GN14]). Let (M, g) be a 3-dimensional Riemannian manifold. Show that the curvature tensor is determined by the Ricci curvature tensor. More precisely, assume that R, R' are covariant 4-tensors satisfying the same identities as the curvature tensor and let $\mathrm{Ric}, \mathrm{Ric}'$ be given by $\mathrm{Ric}_{ij} = R_{kijk}$, $\mathrm{Ric}'_{ij} = R'_{kijk}$. Show that if $\mathrm{Ric} = \mathrm{Ric}'$ then $R = R'$.

Solution. Consider the identities satisfied by the curvature tensor, as well as the identity saying that the trace is 0:

$$\begin{aligned} R_{ijkl} &= -R_{jikl} && \text{(2a)}\\ R_{ijkl} &= R_{klij} && \text{(2b)}\\ R_{ijkl} + R_{jkil} + R_{kijl} &= 0 && \text{(2c)}\\ R_{kijk} &= 0. && \text{(2d)} \end{aligned}$$

Define vector spaces

$$\begin{aligned} V &= \{R \mid R \text{ is a covariant 4-tensor satisfying (2a), (2b), (2c)}\}\\ W &= \{S \mid S \text{ is a symmetric covariant 2-tensor}\} \end{aligned}$$

and a linear map $\phi\colon V \longrightarrow W$ given by $(\phi(R))_{ij} = R_{kijk}$. With this language, what we wish to show is that ϕ is injective. Define

$$\begin{aligned} U &= \ker \phi\\ &= \{R \mid R \text{ is a covariant 4-tensor satisfying (2a), (2b), (2c), (2d)}\}. \end{aligned}$$

We will compute the dimension of U as a function of the dimension of M, and we will see that if $\dim M = 3$ then $\dim U = 0$. Denote $n = \dim M$ and

$$\begin{aligned} X &\coloneqq \{R \mid R \text{ is a covariant 4-tensor satisfying (2a), (2b)}\},\\ N_3 &\coloneqq \text{number of equations in (2c) independent from (2a), (2b)},\\ N_4 &\coloneqq \text{number of equations in (2d) independent from (2a), (2b)}. \end{aligned}$$

Then

$$\dim U = \dim X - N_3 - N_4.$$

We compute $\dim X$. For this, let R be an element of X and write R as an $n^2 \times n^2$ matrix

$$\begin{bmatrix} R_{1111} & R_{1112} & \cdots & R_{11nn}\\ R_{1211} & R_{1212} & \cdots & R_{12nn}\\ \vdots & \vdots & \ddots & \vdots\\ R_{nn11} & R_{nn12} & \cdots & R_{nnnn} \end{bmatrix}.$$

(2b) says that this matrix is symmetric and (2a) says that in this matrix, each row and column assembles into an $n \times n$ antisymmetric matrix. The number of independent components of an $n \times n$ antisymmetric matrix is

$$m = \frac{n(n-1)}{2}.$$

So, $\dim X$ is equal to the number of independent components of an $m \times m$ symmetric matrix, that is

$$\dim X = \frac{m(m+1)}{2} = \frac{1}{2}\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + 1\right).$$

We compute N_3. For each tuple of 4 indices $i, j, k, l \in \{1, \ldots, n\}$, (2c) gives us an equation. However, some of these equations will be linearly dependent.

- If the first 3 indices are permuted evenly, we get the same equation. For example:

$$\begin{aligned} ijkl\colon \quad & 0 = R_{ijkl} + R_{jkil} + R_{kijl} \\ jkil\colon \quad & 0 = R_{jkil} + R_{kijl} + R_{ijkl} \end{aligned}$$

- If the fist 3 indices are permuted oddly, we get the same equation. For example:

$$\begin{aligned} ijkl\colon \quad & 0 = R_{ijkl} + R_{jkil} + R_{kijl} \\ jikl\colon \quad & 0 = R_{jikl} + R_{ikjl} + R_{kjil} \\ & \quad = -R_{ijkl} - R_{kijl} - R_{jkil} \end{aligned}$$

- If we permute the last and second last indices, we get the same equation:

$$\begin{aligned} ijkl\colon \quad & 0 = R_{ijkl} + R_{jkil} + R_{kijl} \\ ijlk\colon \quad & 0 = R_{ijlk} + R_{jlik} + R_{lijk} \\ & \quad = -R_{ijkl} + R_{ikjl} + R_{jkli} \\ & \quad = -R_{ijkl} - R_{kijl} - R_{jkil} \end{aligned}$$

Therefore, we get a different equation for each unordered set of indices $\{i, j, k, l\}$, and

$$N_3 = \begin{cases} \binom{n}{4} & \text{if } n \geq 3 \\ 0 & \text{if } n < 3 \end{cases}.$$

We compute N_4. For each tuple of indices $i, j \in \{1, \ldots, n\}$, we get an equation $R_{kijk} = 0$. But, the equations for ij and those for ji are the same:

$$R_{kijk} = R_{jkki} = -R_{kjki} = R_{kjik}.$$

Therefore,

$$N_4 = \frac{n(n+1)}{2}.$$

Then, if $n \geq 3$

$$\begin{aligned} \dim U &= \dim X - N_3 - N_4 \\ &= \frac{1}{2}\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + 1\right) - \binom{n}{4} - \frac{n(n+1)}{2} \\ &= \frac{1}{12} n(n+1)(n+2)(n-3) \end{aligned}$$

and $\dim U = 0$ if $n = 3$. $\square$

Exercise 9.4 (totally geodesic submanifolds of model spaces)**.**

(a) Let $M \subset (S^n, g_{S^n})$ be an embedded submanifold of dimension m. Show that M is a connected complete totally geodesic submanifold if and only if there exists $\varphi \in \operatorname{Isom}(S^n, g_{S^n})$ such that $\varphi(M) = S^m := \{(x^1, ..., x^{n+1}) \in S^n \mid x^{m+2} = \cdots = x^{n+1} = 0\}$.

(b) Let $M \subset (\mathbb{H}^n, g_{\mathbb{H}^n})$ be an embedded submanifold of dimension m. Show that M is a connected complete totally geodesic submanifold if and only if there exists $\varphi \in \operatorname{Isom}(\mathbb{H}^n, g_{\mathbb{H}^n})$ such that $\varphi(M) = \mathbb{H}^m := \{(x^1, ..., x^n) \in \mathbb{H}^n \mid x^m = \cdots = x^{n-1} = 0\}$.

Solution. (a): We show that $S^m \subset S^n$ is totally geodesic. To see this, it suffices to assume that $\gamma\colon I \longrightarrow S^m$ is a geodesic in S^m and to prove that $\iota \circ \gamma\colon I \longrightarrow S^n$ is a geodesic in S^n.

$$\begin{aligned}
&\gamma \text{ is a geodesic on } S^m \\
&\quad \Longleftrightarrow \forall t \in I\colon \ddot{\gamma}(t) \perp S^m \\
&\quad \Longleftrightarrow \forall t \in I\colon \ddot{\gamma}(t) \text{ is proportional to } \gamma(t) \\
&\quad \Longleftrightarrow \forall t \in I\colon (\iota \circ \gamma)''(t) \perp S^n \\
&\quad \Longleftrightarrow \iota \circ \gamma \text{ is a geodesic on } S^n.
\end{aligned}$$

Implication $(\Longleftarrow)$ now follows because M is isometric to the connected, complete, totally geodesic submanifold S^m.

We prove the implication $(\Longrightarrow)$. Choose $p \in M$ and $x \in S^m$. Recall that the group of isometries of $(S^n_R, g_{S^n_R})$ acts transitively (proven in the lecture notes) on the set

$$\mathcal{S}^n_R = \{(p, E_1, \ldots, E_n) \mid p \in S^n_R, E_1, \ldots, E_n \text{ is an orthonormal basis of } T_p S^n_R\}.$$

Then, there exists $\varphi \in \operatorname{Isom}(S^n, g_{S^n})$ such that $\varphi(p) = x$ and $\mathrm{D}\varphi(p) T_p M = T_x S^m$. So, $\varphi(M)$ and S^m are connected, complete, totally geodesic submanifolds and $T_x \varphi(M) = T_x S^m$. This implies that $\varphi(M) = S^m$.

(b): We show that $\mathbb{H}^m \subset \mathbb{H}^n$ is totally geodesic. To see this, it suffices to assume that $\gamma\colon I \longrightarrow \mathbb{H}^m$ is a geodesic in $\mathbb{H}^m$ and to prove that $\iota \circ \gamma\colon I \longrightarrow \mathbb{H}^n$ is a geodesic in $\mathbb{H}^n$.

$$\begin{aligned}
&\gamma \text{ is a geodesic on } \mathbb{H}^m \\
&\quad \Longleftrightarrow \gamma = \text{ is a vertical half-line or a semicircle w. centre on } \{x^m = 0\} \\
&\quad \Longrightarrow \iota \circ \gamma = \text{ is a vertical half-line or a semicircle w. centre on } \{x^n = 0\} \\
&\quad \Longleftrightarrow \iota \circ \gamma \text{ is a geodesic on } \mathbb{H}^n_R.
\end{aligned}$$

The remainder of the solution is analogous. Implication $(\Longleftarrow)$ now follows because M is isometric to the connected, complete, totally geodesic submanifold $\mathbb{H}^m$.

We prove the implication $(\Longrightarrow)$. Choose $p \in M$ and $x \in \mathbb{H}^m$. Recall that the group of isometries of $(\mathbb{H}^n_R, g_{\mathbb{H}^n_R})$ acts transitively (proven in the lecture notes) on the set

$$\mathcal{H}^n_R = \{(p, E_1, \ldots, E_n) \mid p \in \mathbb{H}^n_R, E_1, \ldots, E_n \text{ is an orthonormal basis of } T_p \mathbb{H}^n_R\}.$$

Then, there exists $\varphi \in \operatorname{Isom}(\mathbb{H}^n, g_{\mathbb{H}^n})$ such that $\varphi(p) = x$ and $\mathrm{D}\varphi(p) T_p M = T_x \mathbb{H}^m$. So, $\varphi(M)$ and $\mathbb{H}^m$ are connected, complete, totally geodesic submanifolds and $T_x \varphi(M) = T_x \mathbb{H}^m$. This implies that $\varphi(M) = \mathbb{H}^m$. $\square$

10 Exercise sheet No. 10 - 28-01-2021

Exercise 10.1 (sectional curvature of model spaces)**.**

(a) Show that $(\mathbb{R}^n, g_{\mathbb{R}^n})$ has constant sectional curvature $\kappa^{\mathbb{R}^n} = 0$.

(b) Show that $(S^n_R, g_{S^n_R})$ has constant sectional curvature $\kappa^{S^n_R} = 1/R^2$, using the following facts:

- The group of isometries of $(S^n_R, g_{S^n_R})$ acts transitively (proven in the lecture notes) on the set
$$\mathcal{S}^n_R = \{(p, E_1, \dots, E_n) \mid p \in S^n_R, E_1, \dots, E_n \text{ is an orthonormal basis of } T_p S^n_R\}.$$
- The maps $\iota\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^{n+1}$, $\iota\colon S^2_R \longrightarrow S^n_R$ given by $\iota(x, y, z) = (x, y, 0, \dots, 0, z)$ are isometric embeddings and $S^2_R \longrightarrow S^n_R$ is totally geodesic.
- The metric and nonzero Christoffel symbols of S^2_R are given in spherical coordinates by
$$\begin{aligned} g &= R^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\varphi \otimes \mathrm{d}\varphi) \\ \Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta \\ \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} &= \frac{\cos\theta}{\sin\theta}. \end{aligned}$$

(c) Show that $(\mathbb{H}^n, g_{\mathbb{H}^n})$ has constant sectional curvature $\kappa^{\mathbb{H}^n} = -1/R^2$, using the same reasoning as above. The metric and nonzero Christoffel symbols of $(\mathbb{H}^n, g_{\mathbb{H}^n})$ are given by:
$$\begin{aligned} g &= \frac{R^2}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y) \\ \Gamma^x_{xy} = \Gamma^x_{yx} = -\Gamma^y_{xx} = \Gamma^y_{yy} &= -\frac{1}{y}. \end{aligned}$$

Solution. (a):
$$\begin{aligned} \Gamma^k_{ij} &= 0 && [\text{Levi-Civita connection of } \mathbb{R}^n] \\ &\implies R_{ijkl} && [\text{formula for } R_{ijkl} \text{ in local coordinates}] \\ &\implies \kappa = 0 && [\text{definition of } \kappa]. \end{aligned}$$

(b): Denote by N the north pole of S^2_R and S^n_R and define $\sigma = \operatorname{im} \mathrm{D}\iota(N)$. Since $\mathrm{Isom}(S^n_R, g_{S^n_R})$ acts transitively on $\mathcal{S}^n_R$, $(S^n_R, g_{S^n_R})$ has constant curvature $\kappa^{S^n_R} = \kappa^{S^n_R}_N(\sigma)$. Since $S^2_R \longrightarrow S^n_R$ is totally geodesic, the second fundamental form of $S^2_R \longrightarrow S^n_R$ is zero. Therefore, the Levi-Civita connection of S^2_R is the restriction of that of S^n_R, and analogously for the curvature tensors and the scalar curvature. So, we can compute $\kappa^{S^n_R}$ as follows:
$$\begin{aligned} \kappa^{S^n_R} &= \kappa^{S^n_R}_N(\sigma) \\ &= \kappa^{S^2_R}_N(T_N S^2_R) \end{aligned}$$

$$
\begin{aligned}
&= \frac{R_{\theta\varphi\varphi\theta}}{\|\partial_\theta\|^2\|\partial_\varphi\|^2}\\
&= \frac{1}{\|\partial_\theta\|^2\|\partial_\varphi\|^2} g_{\alpha\theta}\Big(\partial_\theta\Gamma^\alpha_{\varphi\varphi} - \partial_\varphi\Gamma^\alpha_{\theta\varphi} + \Gamma^\beta_{\varphi\varphi}\Gamma^\alpha_{\theta\beta} - \Gamma^\beta_{\theta\varphi}\Gamma^\alpha_{\varphi\beta}\Big)\\
&= \frac{1}{\|\partial_\theta\|^2\|\partial_\varphi\|^2} g_{\theta\theta}\Big(\partial_\theta\Gamma^\theta_{\varphi\varphi} - \partial_\varphi\Gamma^\theta_{\theta\varphi} + \Gamma^\theta_{\varphi\varphi}\Gamma^\theta_{\theta\theta} - \Gamma^\varphi_{\theta\varphi}\Gamma^\theta_{\varphi\varphi}\Big)\\
&= \frac{1}{R^2R^2\sin^2\theta} R^2\Big(\frac{\partial}{\partial\theta}(-\sin\theta\cos\theta) + \sin\theta\cos\theta\frac{\cos\theta}{\sin\theta}\Big)\\
&= \frac{1}{R^2\sin^2\theta}\Big(-\cos^2\theta + \sin^2\theta + \cos^2\theta\Big)\\
&= \frac{1}{R^2}.
\end{aligned}
$$

(c): Recall that

$$
\mathbb{H}^n_R = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}
$$

and $(g_{\mathbb{H}^n_R})_{ij} = \frac{R^2}{x_n^2}\delta_{ij}$.

The map $\iota\colon \mathbb{H}^2_R \longrightarrow \mathbb{H}^n_R$ given by $\iota(x,y) = (x, 0, \ldots, 0, y)$ is an isometric embedding. Let N denote the north pole for $\mathbb{H}^2$ and $\mathbb{H}^n$, $\sigma = \operatorname{im} \mathrm{D}\iota(N)$ and $\mathbb{H}_\sigma = \operatorname{im}\iota$. Since $\operatorname{Isom}(\mathbb{H}^n_R, g_{\mathbb{H}^n_R})$ acts transitively (proven in the lecture notes) on

$$
\mathcal{H}^n_R = \{(p, E_1, \ldots, E_n) \mid p \in \mathbb{H}^n_R, E_1, \ldots, E_n \text{ is an orthonormal basis of } T_p\mathbb{H}^n_R\},
$$

$(\mathbb{H}^n_R, g_{\mathbb{H}^n_R})$ has constant curvature $\kappa^{\mathbb{H}^n_R} = \kappa_N^{\mathbb{H}^n_R}(\sigma)$.

Since $\mathbb{H}^2_R \longrightarrow \mathbb{H}^n_R$ is totally geodesic, the second fundamental form of $\mathbb{H}^2_R \longrightarrow \mathbb{H}^n_R$ is zero. Therefore, the Levi-Civita connection of $\mathbb{H}^2_R$ is the restriction of that of $\mathbb{H}^n_R$, and analogously for the curvature tensors and the scalar curvature. So, we can compute $\kappa^{\mathbb{H}^n_R}$ as follows:

$$
\begin{aligned}
\kappa^{\mathbb{H}^n_R} &= \kappa_N^{\mathbb{H}^n_R}(\sigma)\\
&= \kappa_N^{\mathbb{H}^2_R}(T_N\mathbb{H}^2_R)\\
&= \frac{R_{xyyx}}{\|\partial_x\|^2\|\partial_y\|^2}\\
&= \frac{1}{\|\partial_x\|^2\|\partial_y\|^2} g_{\alpha x}\Big(\partial_x\Gamma^\alpha_{yy} - \partial_y\Gamma^\alpha_{xy} + \Gamma^\beta_{yy}\Gamma^\alpha_{x\beta} - \Gamma^\beta_{xy}\Gamma^\alpha_{y\beta}\Big)\\
&= \frac{1}{\|\partial_x\|^2\|\partial_y\|^2} g_{xx}\Big(\partial_x\Gamma^x_{yy} - \partial_y\Gamma^x_{xy} + \Gamma^y_{yy}\Gamma^x_{xy} - \Gamma^x_{xy}\Gamma^x_{yx}\Big)\\
&= \frac{1}{\frac{R^2}{y^2}\frac{R^2}{y^2}} \frac{R^2}{y^2}\Big(\frac{\partial}{\partial y}\Big(-\frac{1}{y}\Big) + \Big(-\frac{1}{y}\Big)\Big(-\frac{1}{y}\Big) - \Big(-\frac{1}{y}\Big)\Big(-\frac{1}{y}\Big)\Big)\\
&= \frac{y^2}{R^2}\Big(-\frac{1}{y^2}\Big)\\
&= -\frac{1}{R^2}.
\end{aligned}
$$

□

Exercise 10.2. Let (M, g) be a 2-dimensional Riemannian manifold. Show that for every $p \in M$ we have that $\mathcal{S}(p) = 2\kappa_p$.

Solution. Let $p \in M$ and $\{E_1, E_2\}$ be an orthonormal basis of T_pM. Then,

$$\begin{aligned}
\mathcal{S}(p) &= \operatorname{Ric}(E_1, E_1) + \operatorname{Ric}(E_2, E_2) && [\text{by definition of } \mathcal{S}]\\
&= \mathcal{R}(E_1, E_1, E_1, E_1) + \mathcal{R}(E_2, E_1, E_1, E_2) && [\text{by definition of Ric}]\\
&\quad + \mathcal{R}(E_1, E_2, E_2, E_1) + \mathcal{R}(E_2, E_2, E_2, E_2) &&\\
&= \mathcal{R}(E_2, E_1, E_1, E_2) + \mathcal{R}(E_1, E_2, E_2, E_1) && [\text{identities of } \mathcal{R}]\\
&= 2\mathcal{R}(E_1, E_2, E_2, E_1) && [\text{identities of } \mathcal{R}]\\
&= 2\frac{\mathcal{R}(E_1, E_2, E_2, E_1)}{\|E_1\|^2\|E_2\|^2 - \langle E_1, E_2\rangle^2} && [\{E_1, E_2\} \text{ is orthonormal}]\\
&= 2\kappa(p) && [\text{by definition of } \kappa]. \qquad \square
\end{aligned}$$

Exercise 10.3. Let (N, h) be a Riemannian manifold and $f\colon N \longrightarrow \mathbb{R}$ be a smooth function such that 0 is a regular value of f, i.e. $\nabla f(p) \neq 0$ for any $p \in f^{-1}(0)$. Since 0 is a regular value of f, $M := f^{-1}(0)$ is a smooth hypersurface of N. Show that:

(a) The vector field $n = \frac{\nabla f}{\|\nabla f\|}$ is a unit normal vector field defined in a neighbourhood of M and $L(X) := L(n, X) = \frac{1}{\|\nabla f\|}(\nabla_X \nabla f)^\top$.

(b) $\mathfrak{b}_n(X, Y) = -\frac{1}{\|\nabla f\|}\langle \nabla_X \nabla f, Y\rangle = -\frac{1}{\|\nabla f\|}\operatorname{Hess}(f)(X, Y)$.

(c) If X, Y are orthonormal, then

$$\kappa^M(X, Y) = \kappa^N(X, Y) + \frac{1}{\|\nabla f\|^2}\det\begin{pmatrix}\operatorname{Hess}(f)(X, X) & \operatorname{Hess}(f)(X, Y)\\ \operatorname{Hess}(f)(X, Y) & \operatorname{Hess}(f)(Y, Y)\end{pmatrix}.$$

Solution. (a): The vector $n = \frac{\nabla f}{\|\nabla f\|}$ is well defined in a neighbourhood of M because $\nabla f(p) \neq 0$ for every $p \in M$. Also, n has unit norm. We show that ∇f is normal to M. For this, it suffices to assume that $X \in \mathfrak{X}(M)$ and to prove that $\langle \nabla f, X\rangle = 0$:

$$\begin{aligned}
\langle \nabla f, X\rangle &= \mathrm{d}f(X) && [\text{by definition of gradient}]\\
&= X(f) && [\text{by definition of exterior derivative}]\\
&= 0 && [M = f^{-1}(0) \text{ and } X \in \mathfrak{X}(M)].
\end{aligned}$$

We prove the formula for L:

$$\begin{aligned}
L(n, X) &= (\nabla_X n)^\top && [\text{by definition of } L(n, X)]\\
&= \left(\nabla_X\left(\frac{\nabla f}{\|\nabla f\|}\right)\right)^\top && [\text{by definition of } n]\\
&= \left(X\left(\frac{1}{\|\nabla f\|}\right)\nabla f + \frac{1}{\|\nabla f\|}\nabla_X \nabla f\right)^\top && [\text{Leibniz rule}]\\
&= X\left(\frac{1}{\|\nabla f\|}\right)\nabla f^\top + \frac{1}{\|\nabla f\|}(\nabla_X \nabla f)^\top &&\\
&= \frac{1}{\|\nabla f\|}(\nabla_X \nabla f)^\top && [\nabla f \text{ is normal to } M].
\end{aligned}$$

(b): We prove the first formula for $\mathfrak{b}_n$:

$$\mathfrak{b}_n(X, Y) = -\langle L(n, X), Y\rangle \qquad [\text{definitions of } L_n \text{ and } \mathfrak{b}_n]$$

$$\begin{aligned}
&= -\frac{1}{\|\nabla f\|}\langle(\nabla_X \nabla f)^\top, Y\rangle && \text{[by (a)].}\\
&= -\frac{1}{\|\nabla f\|}\langle \nabla_X \nabla f, Y\rangle && \text{[Y is tangent to M].}
\end{aligned}$$

We prove the formula for the Hessian:

$$\begin{aligned}
\operatorname{Hess}(f)(X,Y) &\coloneqq (\nabla \mathrm{d}f)(X,Y) && \text{[by definition of Hessian]}\\
&= (\nabla_X \mathrm{d}f)(Y) && \text{[definition of total covariant derivative]}\\
&= \nabla_X(\mathrm{d}f(Y)) - \mathrm{d}f(\nabla_X Y) && \text{[covariant derivative of a form]}\\
&= \nabla_X\langle \nabla f, Y\rangle - \langle \nabla f, \nabla_X Y\rangle && \text{[by definition of gradient]}\\
&= \langle \nabla_X \nabla f, Y\rangle && \text{[∇ is compatible with h].}
\end{aligned}$$

(c):

$$\begin{aligned}
&\kappa^M(X,Y) - \kappa^N(X,Y)\\
&= \frac{\langle B(X,X), B(Y,Y)\rangle - \|B(X,Y)\|^2}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} && \text{[proven in the lecture notes]}\\
&= \langle B(X,X), B(Y,Y)\rangle - \|B(X,Y)\|^2 && \text{[X,Y are orthonormal]}\\
&= \langle \mathfrak{b}(X,X)n, \mathfrak{b}(Y,Y)n\rangle - \|\mathfrak{b}(X,Y)n\|^2 && \text{[$(T_pM)^\perp$ is 1-dimensional]}\\
&= \mathfrak{b}(X,X)\mathfrak{b}(Y,Y) - \mathfrak{b}(X,Y)^2\\
&= \det\begin{pmatrix} \mathfrak{b}(X,X) & \mathfrak{b}(X,Y)\\ \mathfrak{b}(X,Y) & \mathfrak{b}(Y,Y)\end{pmatrix} && \text{[by definition of determinant]}\\
&= \frac{1}{\|\nabla f\|^2}\det\begin{pmatrix} \operatorname{Hess}(f)(X,X) & \operatorname{Hess}(f)(X,Y)\\ \operatorname{Hess}(f)(X,Y) & \operatorname{Hess}(f)(Y,Y)\end{pmatrix} && \text{[by (b)].}
\end{aligned}$$

□

11 Exercise sheet No. 11 - 04-02-2021

Exercise 11.1 (Local Uniqueness of Constant Curvature Metrics)**.** Let (M, g) and (N, h) be Riemannian manifolds with constant sectional curvature C. Show that M and N are locally isometric, i.e. that for any $p \in M$, $q \in N$, there exist neighbourhoods U of p in M and V of q in N and an isometry $f\colon U \longrightarrow V$.

Solution. Choose (U, ϕ), (V, ψ) normal coordinate charts around p and q respectively, such that $O := \phi(U) = \psi(V) \subset \mathbb{R}^n$ (this condition can be arranged by resizing the sets U and V). Consider the following commutative diagram:

$$\begin{tikzcd}
(M,g) & (U,g) \arrow[l, hook'] \arrow[r, "\phi"] \arrow[d, "f:=\psi^{-1}\circ\mathrm{id}_O\circ\phi"'] & (O,(\phi^{-1})^*g) \arrow[r, hook] \arrow[d, "\mathrm{id}_O"] & \mathbb{R}^n \\
(N,h) & (V,h) \arrow[l, hook'] \arrow[r, "\psi"'] & (O,(\psi^{-1})^*h) \arrow[r, hook] & \mathbb{R}^n
\end{tikzcd}.$$

By the proposition describing the metric (of a manifold with constant sectional curvature) with respect to normal coordinates, $\mathrm{id}_O\colon (O, (\phi^{-1})^*g) \longrightarrow (O, (\psi^{-1})^*h)$ is an isometry and $f\colon \psi^{-1} \circ \mathrm{id}_O \circ\phi\colon (U, g) \longrightarrow (V, h)$ is an isometry. $\square$

Exercise 11.2. Let (M, g) be a Riemannian manifold, $p \in M$ and $\gamma : [0, a] \to M$ be a geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = V \in T_pM$. Let $W \in T_pM$ be a unit norm vector and let J be the Jacobi field along γ with $J(0) = 0$ and $(\mathrm{D}_tJ)(0) = W$. Show that:

(a) $\|J(t)\|^2_{\gamma(t)}$ can be approximated near $t = 0$ by

$$\|J(t)\|^2_{\gamma(t)} = t^2 - \frac{t^4}{3} g_p(R(W, V)V, W) + O(t^5).$$

(b) If γ is parametrized by arc-length, V, W are orthogonal, and $\sigma = \mathrm{span}\{V, W\}$, then we can approximate $\|J(t)\|^2_{\gamma(t)}$ by

$$\begin{aligned}
\|J(t)\|^2_{\gamma(t)} &= t^2 - \frac{t^4}{3}\kappa_p(\sigma) + O(t^5),\\
\|J(t)\|_{\gamma(t)} &= t - \frac{t^3}{6}\kappa_p(\sigma) + O(t^4).
\end{aligned}$$

Solution. (a): We recall that if $\xi \in T_pM$, $\gamma(t) = \exp_p(t\xi)$ and Y, Z are Jacobi fields along γ such that $Y(0) = 0$, $(\mathrm{D}_tY)(0) = \eta$ and $Z(0) = 0$, $(\mathrm{D}_tZ)(0) = \zeta$, then

$$\langle Y(t), Z(t)\rangle_{\gamma(t)} = t^2\langle \eta, \zeta\rangle_p - \frac{t^4}{3}\langle R(\eta, \xi)\xi, \zeta\rangle_p + O(t^5).$$

Applying this expression with $\xi = V$, $\eta = \zeta = W$ and $Y = Z = J$ we conclude that

$$\begin{aligned}
\|J(t)\|^2_{\gamma(t)} &= t^2\|W\|^2_p - \frac{t^4}{3}\langle R(W, V)V, W\rangle_p + O(t^5)\\
&= t^2 - \frac{t^4}{3}\langle R(W, V)V, W\rangle_p + O(t^5).
\end{aligned}$$

(b): We compute the sectional curvature at p with respect to $\sigma = \operatorname{span}\{V, W\}$:

$$\begin{aligned}
\kappa_p(\sigma) &= \frac{R(W,V,V,W)}{\|W\|^2\|V\|^2 - \langle W,V\rangle^2} && \text{[by definition of } \kappa_p(\sigma)\text{]}\\
&= R(W,V,V,W) && \text{[} \{V,W\} \text{ is an orthonormal basis of } \sigma\text{]}\\
&= \langle R(W,V)V, W\rangle && \text{[by definition of } R(\cdot,\cdot,\cdot,\cdot)\text{]}.
\end{aligned}$$

Therefore, by (a),

$$\begin{aligned}
\|J(t)\|^2_{\gamma(t)} &= t^2 - \frac{t^4}{3}\langle R(W,V)V,W\rangle_p + O(t^5)\\
&= t^2 - \frac{t^4}{3}\kappa_p(\sigma) + O(t^5).
\end{aligned}$$

Let now $f(t) = t - \frac{t^3}{6}\kappa_p(\sigma) + O(t^4)$. Then,

$$\begin{aligned}
f^2(t) &= \left(t - \frac{t^3}{6}\kappa_p(\sigma) + O(t^4)\right)^2\\
&= t^2 - \frac{t^4}{3}\kappa_p(\sigma) + O(t^5)\\
&= \|J(t)\|^2_{\gamma(t)}.
\end{aligned}$$

Therefore $\|J(t)\|_{\gamma(t)} = f(t) = t - \frac{t^3}{6}\kappa_p(\sigma) + O(t^4)$. ☐

Exercise 11.3 (Radial Gauss Lemma)**.** Let (M, g) be a Riemannian manifold, $p \in M$, $W, V \in T_pM$, and $\gamma(t) = \exp_p(tV)$. Show that

$$\Big\langle \mathrm{D}\exp_p(tV)(W), \mathrm{D}\exp_p(tV)V \Big\rangle_{\gamma(t)} = \Big\langle W, V \Big\rangle_p.$$

Solution. There exists a unique $\lambda \in \mathbb{R}$ and $U \in T_pM$ orthogonal to V such that $W = U + \lambda V$. Let X, Y, Z be the Jacobi vector fields along γ with initial conditions

$$\begin{aligned}
X(0) &= 0, & (\mathrm{D}_t X)(0) &= U,\\
Y(0) &= 0, & (\mathrm{D}_t Y)(0) &= \lambda V,\\
Z(0) &= 0, & (\mathrm{D}_t Z)(0) &= W.
\end{aligned}$$

Notice that $Z = X + Y$. By the formula for the derivative of the exponential map,

$$\begin{aligned}
\mathrm{D}\exp_p(tV)W &= Z(t),\\
\mathrm{D}\exp_p(tV)V &= \frac{1}{\lambda}Y(t).
\end{aligned}$$

$\mathrm{D}\exp_p(tV)V$ can also be computed as

$$\begin{aligned}
\mathrm{D}\exp_p(tV)V &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \exp_p(tV + sV) && \text{[by definition of derivative]}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \gamma(t+s) && \text{[by definition of } \gamma\text{]}\\
&= \dot\gamma(t).
\end{aligned}$$

Then,

$$\begin{aligned}
\left\langle \mathrm{D}\exp_p(tV)(W), \mathrm{D}\exp_p(tV)V \right\rangle_{\gamma(t)} &= \frac{1}{\lambda}\langle Z(t), Y(t)\rangle_{\gamma(t)} \\
&= \frac{1}{\lambda}\langle X(t) + Y(t), Y(t)\rangle_{\gamma(t)} \\
&= \frac{1}{\lambda}\langle Y(t), Y(t)\rangle_{\gamma(t)} \\
&= \lambda\|\dot\gamma(t)\|^2_{\gamma(t)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\langle \mathrm{D}\exp_p(tV)(W), \mathrm{D}\exp_p(tV)V \right\rangle_{\gamma(t)} \\
&\quad = \lambda\|\dot\gamma(t)\|^2_{\gamma(t)} && \text{[by the computation above]} \\
&\quad = \lambda\|\dot\gamma(0)\|^2_{\gamma(t)} && \text{[γ is a geodesic]} \\
&\quad = \left\langle \mathrm{D}\exp_p(0)(W), \mathrm{D}\exp_p(0)V \right\rangle_{\gamma(0)} && \text{[by the computation above]} \\
&\quad = \langle W, V\rangle_{\gamma(0)}. && \text{[$\mathrm{D}\exp_p(0) = \mathrm{id}$].}
\end{aligned}$$

□

Exercise 11.4 (boundary problems for Jacobi fields)**.** Let (M, g) be a Riemannian manifold and $\gamma\colon [0,1] \longrightarrow M$ be a geodesic. Show that the two-point boundary problem for Jacobi fields admits a unique solution for every pair of vectors $X \in T_{\gamma(0)}M$ and $Y \in T_{\gamma(1)}M$ if and only if $\gamma(0)$ and $\gamma(1)$ are not conjugate along γ.

Solution. Define $p = \gamma(0)$, $q = \gamma(1)$ and $V = \dot\gamma(0)$, so that $\gamma(t) = \exp_p(tV)$. Define the notions of one point boundary problem and two point boundary problem: a vector field J along γ is a solution of

- the one point boundary problem with conditions $X \in T_{\gamma(0)}M$ and $Z \in T_V(T_{\gamma(0)}M)$ if J is Jacobi, $J(0) = X$ and $(\mathrm{D}_t J)(0) = Z$;
- the two point boundary problem with conditions $X \in T_{\gamma(0)}M$ and $Y \in T_{\gamma(1)}M$ if J is Jacobi, $J(0) = X$ and $J(1) = Y$.

Recall that the one point boundary problem admits a unique solution for every X, Z. Also recall that

$$\begin{aligned}
&p \text{ and } q \text{ are not conjugate along } \gamma \\
&\quad \Longleftrightarrow \exp_p \text{ is a local diffeomorphism in a neighbourhood of } V \\
&\quad \Longleftrightarrow \mathrm{D}\exp_p(V)\colon T_V(T_pM) \longrightarrow T_qM \text{ is a linear isomorphism.}
\end{aligned}$$

The idea of the proof will be to use the fact that $\mathrm{D}\exp_p(V)$ is a linear isomorphism to translate between the conditions $(\mathrm{D}_t J)(0) = Z \in T_V(T_pM)$ and $J(0) = Y$. This works because of the formula for the derivative of the exponential map.

We show that if $\mathrm{D}\exp_p(V)$ is a linear isomorphism then the two point boundary problem admits a unique solution for every $X \in T_pM$, $Y \in T_qM$. We prove existence, i.e. that there exists a Jacobi vector field J along γ such that $J(0) = X$ and $J(1) = Y$. Define $Z \in T_V(T_pM)$ to be such that $\mathrm{D}\exp_p(V)Z = Y$. Define J to be the solution of the one point boundary problem with conditions $X \in T_{\gamma(0)}M$ and $Z \in T_V(T_{\gamma(0)}M)$. Then, $J(1) = Y$ and J is as desired:

$$\begin{aligned}
J(1) &= \mathrm{D}\exp_p(V) \cdot (\mathrm{D}_t J)(0) && [\text{formula for derivative of exponential}] \\
&= \mathrm{D}\exp_p(V) \cdot Z && [(\mathrm{D}_t J)(0) = Z \text{ by definition of } J] \\
&= Y && [\text{by definition of } Z].
\end{aligned}$$

We prove uniqueness. It suffices to assume that J, J' are solutions of the two point boundary problem with conditions X, Y and to prove that $J = J'$. By uniqueness of solution of the one point boundary problem, it suffices to show that $(\mathrm{D}_t J)(0) = (\mathrm{D}_t J')(0)$. This is true because $\mathrm{D}\exp_p(V)$ is a linear isomorphism and by the formula for the derivative of the exponential:

$$\begin{aligned}
(\mathrm{D}_t J)(0) &= (\mathrm{D}\exp_p(V))^{-1} Y \\
&= (\mathrm{D}_t J')(0).
\end{aligned}$$

We show that if the two point boundary problem admits a unique solution for every $X \in T_pM$, $Y \in T_qM$ then $\mathrm{D}\exp_p(V)$ is a linear isomorphism. It suffices to assume that $Z \in \ker \mathrm{D}\exp_p(V) \subset T_V(T_pM)$ and to prove that $Z = 0$. Define J to be the unique solution of the two point boundary problem with $J(0) = 0$ and $(\mathrm{D}_t J)(0) = Z$. Then,

$$\begin{aligned}
0 &= \mathrm{D}\exp_p(V) Z && [\text{by assumption}] \\
&= J(1) && [\text{by the formula for the derivative of the exponential}].
\end{aligned}$$

Therefore J is a solution of the one point boundary problem with $J(0) = 0$ and $J(1) = 0$. So, by uniqueness of solution of the one point boundary problem, $J = 0$. Therefore $Z = (\mathrm{D}_t J)(0) = 0$. ☐

Exercise 11.5 (energy, from [Gor12])**.** Let (M, g) be a Riemannian manifold. Define the **energy functional**

$$\begin{aligned}
E \colon C^\infty([0,1], M) &\longrightarrow \mathbb{R} \\
\gamma &\longmapsto \frac{1}{2} \int_0^1 \|\dot\gamma(t)\|^2 \mathrm{d}t.
\end{aligned}$$

Let $\gamma \colon [0,1] \longrightarrow M$ be a curve and let $\Gamma \colon (-\varepsilon, \varepsilon) \times [0,1] \longrightarrow M$ be a variation of γ, i.e. $\Gamma(0,t) = \gamma(t)$. Let $V \in C^\infty(\gamma^* TM)$ be given by $V(t) = \frac{\partial \Gamma}{\partial s}(0,t)$. Define $T = \frac{\partial \Gamma}{\partial t}$, $S = \frac{\partial \Gamma}{\partial s}$ and $\gamma_s(t) \coloneqq \Gamma(s,t)$. Prove:

(a) The first variation of energy formula:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} E(\gamma_s) = \langle V, \dot\gamma \rangle \big|_{t=0}^{t=1} - \int_0^1 \langle V, \mathrm{D}_t \dot\gamma \rangle \mathrm{d}t;$$

(b) That γ is a geodesic if and only if $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} E(\gamma_s) = 0$ for every proper variation Γ of γ;

(c) The second variation of energy formula: if γ is a geodesic, then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} E(\gamma_s) = \left\langle \mathrm{D}_s \frac{\partial \Gamma}{\partial s}, \dot\gamma \right\rangle \bigg|_{t=0}^{t=1} + \int_0^1 \Big(\langle R(V, \dot\gamma)V, \dot\gamma \rangle + \|\mathrm{D}_t V\|^2 \Big) \mathrm{d}t.$$

Solution. (a):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^1 \langle\dot\gamma_s,\dot\gamma_s\rangle \mathrm{d}t && \text{[definition of Energy]}\\
&= \frac{1}{2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\langle\dot\gamma_s,\dot\gamma_s\rangle \mathrm{d}t && \text{[differentiation under integral sign]}\\
&= \frac{1}{2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left\langle\frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t}\right\rangle \mathrm{d}t && \text{[definition of } \gamma_s\text{]}\\
&= \int_0^1 \left\langle \mathrm{D}_s\frac{\partial\Gamma}{\partial t},\frac{\partial\Gamma}{\partial t}\right\rangle \mathrm{d}t && \text{[}\nabla\text{ is compatible with } g\text{]}\\
&= \int_0^1 \left\langle \mathrm{D}_t\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle \mathrm{d}t && \text{[symmetry lemma]}\\
&= \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle - \left\langle\frac{\partial\Gamma}{\partial s},\mathrm{D}_t\frac{\partial\Gamma}{\partial t}\right\rangle\right) \mathrm{d}t && \text{[}\nabla\text{ is compatible with } g\text{]}\\
&= \left\langle\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle\Bigg|_{t=0}^{t=1} - \int_0^1 \left\langle\frac{\partial\Gamma}{\partial s},\mathrm{D}_t\frac{\partial\Gamma}{\partial t}\right\rangle \mathrm{d}t && \text{[fundamental theorem of calculus].}
\end{aligned}
$$

At $s=0$, $\frac{\partial\Gamma}{\partial s}(0,t)=V(t)$ and $\frac{\partial\Gamma}{\partial t}(0,t)=\dot\gamma(t)$. Therefore,

$$
\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}E(\gamma_s) = \langle V,\dot\gamma\rangle|_{t=0}^{t=1} - \int_0^1 \langle V,\mathrm{D}_t\dot\gamma\rangle \mathrm{d}t.
$$

(b): The proof is the following string of equivalences:

$$
\begin{aligned}
&\text{for every } \Gamma \text{ a proper variation of } \gamma \text{ we have that } 0 = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}E(\gamma_s)\\
&\quad\Longleftrightarrow \text{for every } \Gamma \text{ a proper variation of } \gamma \text{ we have that } 0 = \langle V,\dot\gamma\rangle|_{t=0}^{t=1} - \int_0^1 \langle V,\mathrm{D}_t\dot\gamma\rangle \mathrm{d}t\\
&\quad\Longleftrightarrow \text{for every } \Gamma \text{ a proper variation of } \gamma \text{ we have that } 0 = \int_0^1 \langle V,\mathrm{D}_t\dot\gamma\rangle \mathrm{d}t\\
&\quad\Longleftrightarrow \mathrm{D}_t\dot\gamma = 0\\
&\quad\Longleftrightarrow \gamma \text{ is a geodesic.}
\end{aligned}
$$

(c):

$$
\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(\gamma_s)\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}s}\int_0^1 \left\langle \mathrm{D}_t\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle \mathrm{d}t\\
&\quad= \int_0^1 \frac{\partial}{\partial s}\left\langle \mathrm{D}_t\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle \mathrm{d}t\\
&\quad= \int_0^1 \left(\left\langle \mathrm{D}_s\mathrm{D}_t\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle + \left\langle \mathrm{D}_t\frac{\partial\Gamma}{\partial s},\mathrm{D}_s\frac{\partial\Gamma}{\partial t}\right\rangle\right) \mathrm{d}t\\
&\quad= \int_0^1 \left(\left\langle \mathrm{D}_t\mathrm{D}_s\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle + \left\langle R(S,T)\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle + \left\|\mathrm{D}_t\frac{\partial\Gamma}{\partial s}\right\|^2\right) \mathrm{d}t\\
&\quad= \int_0^1 \left(\frac{\partial}{\partial t}\left\langle \mathrm{D}_s\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle - \left\langle \mathrm{D}_s\frac{\partial\Gamma}{\partial s},\mathrm{D}_t\frac{\partial\Gamma}{\partial t}\right\rangle + \left\langle R(S,T)\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle + \left\|\mathrm{D}_t\frac{\partial\Gamma}{\partial s}\right\|^2\right) \mathrm{d}t\\
&\quad= \left\langle \mathrm{D}_s\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle\Bigg|_{t=0}^{t=1} + \int_0^1 \left(-\left\langle \mathrm{D}_s\frac{\partial\Gamma}{\partial s},\mathrm{D}_t\frac{\partial\Gamma}{\partial t}\right\rangle + \left\langle R(S,T)\frac{\partial\Gamma}{\partial s},\frac{\partial\Gamma}{\partial t}\right\rangle + \left\|\mathrm{D}_t\frac{\partial\Gamma}{\partial s}\right\|^2\right) \mathrm{d}t.
\end{aligned}
$$

At $s = 0$,

$$
\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}s^2} E(\gamma_s) \\
&\quad = \left\langle \mathrm{D}_s \frac{\partial \Gamma}{\partial s}, \dot{\gamma} \right\rangle \Big|_{t=0}^{t=1} + \int_0^1 \Big(-\left\langle \mathrm{D}_s \frac{\partial \Gamma}{\partial s}, \mathrm{D}_t \dot{\gamma} \right\rangle + \langle R(V, \dot{\gamma})V, \dot{\gamma} \rangle + \|\mathrm{D}_t V\|^2 \Big) \mathrm{d}t \\
&\quad = \left\langle \mathrm{D}_s \frac{\partial \Gamma}{\partial s}, \dot{\gamma} \right\rangle \Big|_{t=0}^{t=1} + \int_0^1 \Big(\langle R(V, \dot{\gamma})V, \dot{\gamma} \rangle + \|\mathrm{D}_t V\|^2 \Big) \mathrm{d}t. \qquad \square
\end{aligned}
$$

12 Exercise sheet No. 12 - 11-02-2021

Review of Homotopy theory. In this exercise sheet, we will need to use some basic facts about homotopy theory, which we now review. Let X be a locally compact Hausdorff space and Y be a connected Hausdorff space. Denote by $C(X,Y)$ the set of continuous maps from X to Y. Denote $I = [0,1]$.

Homotopies, point of view 1. We say that $f, g \in C(X,Y)$ are **homotopic** if there exists a continuous map $H: I \times X \longrightarrow Y$ such that $H(0,\cdot) = f$ and $H(1,\cdot) = g$. In this case, H is a **homotopy** from f to g. "Homotopic" is an equivalence relation on $C(X,Y)$. A **homotopy class** is an equivalence class under this equivalence relation.

Compact-open topology and exponential law. Now, instead of viewing a homotopy as a map $H: I \times X \longrightarrow Y$, we would like to view it as a path of continuous functions $h: I \longrightarrow C(X,Y)$. To do this, we need to give $C(X,Y)$ the structure of a topological space. The **compact-open topology** on $C(X,Y)$ is the topology generated by sets of the form

$$S_{K,U} \coloneqq \{f \in C(X,Y) \mid f(K) \subset U\},$$

for $K \subset X$ compact and $U \subset Y$ open. If Y is a metric space and X is compact, then convergence with respect to the compact-open topology is the same thing as uniform convergence. If in addition X and Y are manifolds, then the compact-open topology also coincides with the C^0-topology. Consider the map

$$\begin{aligned} \Phi: C(I \times X, Y) &\longrightarrow C(I, C(X,Y)) \\ H &\longmapsto h \coloneqq \Phi(H) \end{aligned}$$

where $h(t)(x) = H(t,x)$ for $t \in I$ and $x \in X$. Since X is a locally compact Hausdorff and Y is Hausdorff, then Φ is a homeomorphism (this is a theorem, it's not supposed to be obvious!). This theorem is called the **exponential law** for topological spaces, because it can also be written $Y^{I \times X} \cong (Y^X)^I$.

Homotopies, point of view 2. Using this information, we can now say that for $f, g \in C(X,Y)$ a homotopy from f to g is a path $h: I \longrightarrow C(X,Y)$ from f to g. In this language, f and g are homotopic if and only if they are in the same path component of $C(X,Y)$ and we see that a homotopy class is the same thing as a path connected component of $C(X,Y)$.

Contractible maps. Since Y is connected, all the constant maps $X \longrightarrow Y$ are in the same path component/homotopy class of $C(X,Y)$, which we call the **trivial homotopy class**. A map $X \longrightarrow Y$ is **contractible** if it homotopic to a constant map, i.e. it is an element of the trivial homotopy class. Y is **simply connected** if all elements of $C(S^1,Y)$ are contractible, or equivalently $C(S^1,Y)$ is path connected.

Loops. We will use the above facts with $Y = M$ a manifold and $X = S^1$. In this case, we will say that a homotopy class of maps in $C(X,Y)$ is a free homotopy class of loops. The word "loops" is because $X = S^1$. The word "free" is because the homotopy is a path in $C(X,Y)$ which does not satisfy any other additional assumptions.

Exercise 12.1 (Cartan's lemma, from [Gor12]). Let M be a compact connected Riemannian manifold which is not simply connected and let $\mathcal{C}$ be a nontrivial free homotopy class of loops. Show that there exists a closed geodesic γ in $\mathcal{C}$ such that $l(\gamma) = l \coloneqq \inf_{\eta \in \mathcal{C}} l(\eta)$, as follows:

(a) Show that there exists $\epsilon > 0$ such that for all $p \in M$ we have:

(a.1) for every $x, y \in B(p, \varepsilon/2)$ there exists a unique geodesic $\gamma_{x,y}$ connecting x and y;

(a.2) the map $B(p, \varepsilon/2) \times B(p, \varepsilon/2) \times [0, 1] \longrightarrow M$ given by $(x, y, t) \longmapsto \gamma_{x,y}(t)$ is smooth.

(*Hint: cover M by a finite suitable family of totally normal neighbourhoods*).

(b) Choose $(\eta_j)_j$ a sequence of smooth loops in $\mathcal{C}$ such that $\lim_{j\to+\infty} l(\eta_j) = l = \inf_{\eta\in\mathcal{C}} l(\eta)$ and each $\eta_j : [0, 1] \longrightarrow M$ is parametrized with constant speed (here we used that $C^\infty(S^1, M)$ is a dense subset of $C(S^1, M)$). Define $L = \sup_j l(\eta_j) < +\infty$. Choose a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ with $t_i - t_{i-1} < \varepsilon/2L$ for $i = 1, \ldots, n$. Show that $d(\eta_j(t_{i-1}), \eta_j(t)) < \varepsilon/2$ for every i and $t \in [t_{i-1}, t_i]$.

(c) For each j define γ_j to be the broken geodesic joining $\eta_j(0), \eta_j(t_1), \ldots, \eta_j(1)$ (which we can do by the two previous steps). Since M is compact we can pass to a subsequence (which we denote by the same index j) such that $\gamma_j(t_i)$ converges as $j \to +\infty$ to $p_i \in M$ for every i. Show that $d(p_{i-1}, p_i) < \varepsilon/2$ for every i.

(d) Define γ to be the broken geodesic joining $p_0, \ldots, p_n$. Show that γ is as desired.

Solution. (a): Recall the following fact: for every $p \in M$, there exists a $\delta > 0$ and a neighbourhood U of p such that U is δ-totally normal, i.e.

(a) For all $x, y \in U$ there exists a unique geodesic $\gamma_{x,y}$ from x to y.

(b) The map $U \times U \times [0, 1] \longrightarrow M$ given by $(x, y, t) \longmapsto \gamma_{x,y}(t)$ is smooth.

(c) For all $x \in U$, if $V := \exp_x(B(0, \delta))$ then $\exp_x : B(0, \delta) \longrightarrow V$ is a diffeomorphism.

Use this fact to cover M by finitely many balls $B(p_i, \varepsilon_i/2)$ such that $B(p_i, \varepsilon_i)$ is a δ_i-totally normal ball. Define $\varepsilon = \min_i \varepsilon_i/2$. We show that ε is as desired.

For $p \in M$, we prove (a.1). It suffices to assume that $x, y \in B(p, \varepsilon/2)$ and to prove that there exists a unique geodesic $\gamma_{x,y}$ from x to y. Let i be such that $x \in B(p_i, \varepsilon_i/2)$. Then, $y \in B(p_i, \varepsilon_i)$:

$$\begin{aligned} d(y, p_i) &\le d(y, x) + d(x, p_i) \\ &< \varepsilon + \varepsilon_i/2 \\ &\le \varepsilon_i/2 + \varepsilon_i/2 \\ &= \varepsilon_i. \end{aligned}$$

Since $x, y \in B(p_i, \varepsilon_i)$, which is a δ_i-totally normal ball, there exists a geodesic $\gamma_{x,y}$ from x to y.

For $p \in M$, we prove (a.2). We need to show that $B(p, \varepsilon/2) \times B(p, \varepsilon/2) \times [0, 1] \longrightarrow M$ is smooth. We show that this map is smooth at $(x, y, t) \in B(p, \varepsilon/2) \times B(p, \varepsilon/2) \times [0, 1]$. In a neighbourhood of (x, y, t), this map coincides with the map $B(p_i, \varepsilon_i) \times B(p_i, \varepsilon_i) \times [0, 1] \longrightarrow M$ (from the definition of totally normal), which is smooth.

(b):

$$\begin{aligned}
d(\eta_j(t_{i-1}), \eta_j(t)) &\leq \int_{t_{i-1}}^{t} \|\dot{\eta}_j(s)\| \mathrm{d}s && \text{[by definition of distance]} \\
&\leq \int_{t_{i-1}}^{t_i} \|\dot{\eta}_j(s)\| \mathrm{d}s \\
&= (t_i - t_{i-1}) \|\dot{\eta}_j(0)\| && [\eta_j \text{ is parametrized with constant speed}] \\
&\leq (t_i - t_{i-i}) L && [L = \sup_j l(\eta_j)] \\
&< \varepsilon/2 && [t_i - t_{i-1} < \varepsilon/2L \text{ for } i = 1, \ldots, n].
\end{aligned}$$

(c): By definition of γ_j,

$$d(\gamma_j(t_{i-1}), \gamma_j(t_i)) = d(\eta_j(t_{i-1}), \eta_j(t_i)) < \varepsilon/2.$$

Passing to the limit and using the fact that the distance d is continuous,

$$\lim_{j \to +\infty} d(\gamma_j(t_{i-1}), \gamma_j(t_i)) = d(p_{i-1}, p_i) < \varepsilon/2.$$

(d): We show that γ is in $\mathcal{C}$. First, recall that all the η_j are in $\mathcal{C}$. We now show that all the γ_j are in $\mathcal{C}$. It suffices to show that η_j is homotopic to γ_j. For all $t \in [t_{i-1}, t_i]$, $d(\gamma_j(t), \eta_j(t)) < \varepsilon$:

$$\begin{aligned}
d(\gamma_j(t), \eta_j(t)) &\leq d(\gamma_j(t), \gamma_j(t_{i-1})) + d(\gamma_j(t_{i-1}), \eta_j(t)) \\
&= d(\gamma_j(t), \gamma_j(t_{i-1})) + d(\eta_j(t_{i-1}), \eta_j(t)) \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Using this, we can build a homotopy from $\eta_j|_{[t_{i-1}, t_i]}$ to $\gamma_j|_{t_{i-1}, t_i}$ by using the shortest geodesic from $\gamma_j(t)$ to $\eta_j(t)$. This concludes the proof that all the γ_j are in $\mathcal{C}$. By definition of γ, the γ_j converge to γ (as elements of $C(S^1, M)$ equipped with the compact-open topology). Also, $\mathcal{C} \subset C(S^1, M)$ is a path-connected component. So, γ is in $\mathcal{C}$.

We show that $l(\gamma) = l$.

$$\begin{aligned}
l(\gamma) &= \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \\
&= \lim_{j \to +\infty} \sum_{i=1}^{n} d(\gamma_j(t_{i-1}), \gamma_j(t_i)) \\
&= \lim_{j \to +\infty} \sum_{i=1}^{n} d(\eta_j(t_{i-1}), \eta_j(t_i)) \\
&\leq \lim_{j \to +\infty} \sum_{i=1}^{n} l(\eta_j|_{[t_{i-1}, t_i]}) \\
&= \lim_{j \to +\infty} l(\eta) \\
&= l \\
&= \inf_{\eta \in \mathcal{C}} l(\eta) \\
&\leq l(\gamma).
\end{aligned}$$

We show that γ is a geodesic. It suffices to show that γ is locally length minimizing. Assume by contradiction that γ is not locally length minimizing. Then, there exists a curve $\gamma' \in \mathcal{C}$ such that $l(\gamma') < l(\gamma)$ (γ' is in $\mathcal{C}$ because $\mathcal{C}$ is open and γ' can be taken to be "near" γ). Then,

$$\begin{aligned} l(\gamma') &< l(\gamma) \\ &= \inf_{\eta \in \mathcal{C}} l(\eta) \\ &\leq l(\gamma') \end{aligned}$$

gives us a contradiction. □

Exercise 12.2 (Synge's theorem, from [Gor12])**.** Let M be a Riemannian manifold which is even dimensional, orientable, compact, connected and has positive sectional curvature (at every point $p \in M$, for every 2-dimensional subspace of T_pM). Show that M is simply connected, as follows. Assume by contradiction that there exists $\mathcal{C}$ a nontrivial free homotopy class of loops. By Cartan's lemma, there exists $\gamma\colon [0,1] \longrightarrow M$ a closed geodesic in $\mathcal{C}$ parametrized with constant speed such that $l(\gamma) = \inf_{\eta \in \mathcal{C}} l(\eta)$. Define $p = \gamma(0) = \gamma(1)$, $v = \dot\gamma(0) = \dot\gamma(1)$ and let $P\colon T_pM \longrightarrow T_pM$ be the parallel transport map along γ.

(a) Show that $P\colon T_pM \longrightarrow T_pM$ is orientation preserving, $P(v) = v$ and $P(\langle v\rangle^\perp) = \langle v\rangle^\perp$.

(b) Show that there exists $w \in \langle v\rangle^\perp$ such that $P(w) = w$. (*Hint: consider the canonical form for elements of the orthogonal group*).

(c) Define V to be the vector field along γ given by parallel transporting w along γ. Define Γ to be the variation of γ coming from γ and V and $\gamma_s(t) = \Gamma(s,t)$. Show that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E(\gamma_s) < 0$$

and use these facts to derive a contradiction.

Solution. (a): Consider the map $P_t\colon T_pM \longrightarrow T_{\gamma(t)}M$ defined via parallel transport along γ. This gives us a continuous family of maps $(P_t)_t$ such that $P_0 = \mathrm{id}_{T_pM}$ is orientation preserving and $P_1 = P$. Therefore P is orientation preserving. By definition of geodesic and of parallel transport, P_t maps $v = \dot\gamma(0)$ to $\dot\gamma(t)$, and therefore $P(v) = P(\dot\gamma(0)) = \dot\gamma(1) = \dot\gamma(0) = v$. Since P is an isometry, $P(\langle v\rangle^\perp) = \langle P(v)\rangle^\perp = \langle v\rangle^\perp$.

(b): Let $2n = \dim M$. Choose an orthonormal basis for $\langle v\rangle^\perp$, and consider the induced linear isometry $\langle v\rangle^\perp \longrightarrow \mathbb{R}^{2n-1}$. Use this isometry to view $P\colon \langle v\rangle^\perp \longrightarrow \langle v\rangle^\perp$ as a map $\overline{P}\colon \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^{2n-1}$. Then, $\overline{P}$ is an element of $SO(2n-1) \subset O(2n-1)$. It suffices to show that 1 is an eigenvalue of $\overline{P}$. By the canonical form for elements of

$O(2n-1)$, after an isometric coordinate change we can write $\overline{P}$ as

$$\overline{P} = \begin{pmatrix} R_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & R_k & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (-1)^{j_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & (-1)^{j_l} \end{pmatrix},$$

where $R_1, \dots, R_k$ are 2×2 rotation matrices. Then $2k + l = 2n - 1$ which implies that l is odd. Then, $1 = \det \overline{P} = (-1)^{j_1 + \cdots + j_l}$ which implies that $j_1 + \cdots + j_l$ is even. Since l is odd this implies that there exists $i = 1, \dots, l$ such that $j_i = 0$. Then 1 is an eigenvalue of $\overline{P} \colon \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^{2n-1}$.

(c): We show that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} E(\gamma_s) = 0$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} E(\gamma_s) &= \langle V, \dot{\gamma} \rangle \big|_{t=0}^{t=1} - \int_0^1 \langle V, \mathrm{D}_t \dot{\gamma} \rangle \mathrm{d}t \\ &= 0. \end{aligned}$$

The second derivative can be computed by

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big|_{s=0} E(\gamma_s) &= \left\langle \mathrm{D}_s \frac{\partial \Gamma}{\partial s}, \dot{\gamma} \right\rangle \Big|_{t=0}^{t=1} + \int_0^1 \Big(\langle R(V, \dot{\gamma})V, \dot{\gamma} \rangle + \| \mathrm{D}_t V \|^2 \Big) \mathrm{d}t \\ &= \int_0^1 \langle R(V, \dot{\gamma})V, \dot{\gamma} \rangle \mathrm{d}t \\ &< 0, \end{aligned}$$

where in the last equality we used the fact that M has positive sectional curvature.

So, there exists an $\varepsilon > 0$ such that the function $[0, \varepsilon] \longrightarrow \mathbb{R}$ given by $s \longmapsto E(\gamma_s)$ is strictly decreasing. Then,

$$\begin{aligned} l(\gamma)^2 &\leq l(\gamma_s)^2 && [\gamma \text{ is a geodesic}] \\ &= \left(\int_0^1 \| \dot{\gamma}_s(t) \| \mathrm{d}t \right)^2 && [\text{by definition of length}] \\ &\leq \left(\int_0^1 1 \mathrm{d}t \right) \left(\int_0^1 \| \dot{\gamma}_s(t) \|^2 \mathrm{d}t \right) && [\text{Hölder's inequality}] \\ &= 2E(\gamma_s) && [\text{by definition of Energy}] \\ &< E(\gamma) && [\text{by the discussion above}] \\ &= \int_0^1 \| \dot{\gamma}(t) \|^2 \mathrm{d}t && [\text{by definition of } E(\gamma)] \\ &= l(\gamma)^2 && [\gamma \text{ is parametrized with constant speed}], \end{aligned}$$

which is a contradiction. □

References

[GN14] L. Godinho and J. Natário. *An Introduction to Riemannian Geometry*. Universitext. Springer, first edition, 2014.

[Gor12] Claudio Gorodski. An introduction to Riemannian geometry, 2012.