Bi-Hamiltonian mechanical systems

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Remark (Schouten-Nijenhuis bracket, [Mar97, p. 354], [MM84, p. 146])

Let M be a manifold. For $k \in \mathbb{N}$, define $A^k(M) = C^\infty\Big(M \longleftarrow \bigwedge_{i=1}^k TM\Big)$. Then, the Schouten-Nijenhuis bracket of M is a map $[\cdot,\cdot]\colon A^p(M) \times A^q(M) \longrightarrow A^{p+q-1}(M)$. If $P, Q \in A^2(M)$, we can view them as maps $P, Q\colon \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ and $[P,Q]\colon \Omega^1(M) \times \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ is given by:

$$\begin{aligned} 2[P,Q](\alpha,\beta) = \quad & (L_{P\beta}Q)\alpha + Q(L_{P\alpha}\beta) + Qd(\alpha(P\beta)) \\ & + (L_{Q\beta}P)\alpha + P(L_{Q\alpha}\beta) + Pd(\alpha(Q\beta)). \end{aligned}$$

Definition (Poisson manifold, [MM84, p. 14], [Fer94a, p. 2])

A **Poisson manifold** is a smooth manifold M, equipped with a tensor P of type $(0,2)$ (receives two covectors and outputs a number) such that

(antisymmetric) View P as a map $P\colon \Omega^1(M) \longrightarrow \mathfrak{X}(M)$. Then $P + P^* = 0$.

(Jacobi) $[P, P] = 0$.

P is **nondegenerate** if it has maximal rank (at every point).

Definition (Hamiltonian vector field)

Let (M, P) be a Poisson manifold and $F \in C^\infty(M, \mathbb{R})$ be a function on M. The **Hamiltonian vector field** of F is $X_F := P\mathrm{d}F$.

Definition (bi-Poisson manifold, [Fer94a, p. 3])

A **bi-Poisson manifold** is a smooth manifold M equipped with Poisson structures P and Q such that $[P, Q] = 0$.

Definition (bi-Hamiltonian vector field, [Fer94a, p. 4])

Let (M, P, Q) be a bi-Poisson manifold. Let $F, G \in C^\infty(M, \mathbb{R})$ be such that $P\mathrm{d}F = Q\mathrm{d}G$. Then, the **bi-Hamiltonian vector field** of F, G is the vector field $X = P\mathrm{d}F = Q\mathrm{d}G$.

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In this section, (M, P, Q) is a bi-Poisson manifold with P nondegenerate, such that $\dim M = 2n$.

Definition (recursion operator, [Fer94a, p. 4])

The **recursion operator** of (M, P, Q) is the $(1, 1)$-tensor N given by $N = QP^{-1}\colon TM \longrightarrow TM$.

Definition (T_N = torsion of N, [MM84, p. 14], [Fer94a, p. 4])

We define a tensor T_N of type $(2, 1)$, called the **torsion** of N by $T_N(X, Y) = [NX, NY] - N[NX, Y] - N[X, NY] + N^2[X, Y]$.

Definition (R_N, [MM84, p. 15])

We define a tensor R_N of type $(1, 2)$ by $R_N(\alpha, X) = L_{P\alpha}(N)X - PL_X(N^*\alpha) + PL_{NX}(\alpha)$.

Lemma (eigenvalues of N, [MM84, p. 27])

For each $p \in M$, if λ is an eigenvalue of $N_p \colon T_pM \longrightarrow T_pM$ with eigenspace E, then E is even dimensional. In particular, N_p has at most n distinct eigenvalues.

Definition

N has **nice eigenvalues** if N is diagonalizable and it has eigenvalues $\lambda_1, \dots, \lambda_n$ which are smooth functions, pairwise different at every point, and independent.

Assume from now on (until the end of this section) that N has nice eigenvalues.

Proposition ([Fer94b, p. 11], [MM84, p. 14, 15, 149-155])

N satisfies the following properties:

1. $PN^* = NP$;
2. $T_N = 0$;
3. $R_N = 0$.

Theorem (Magri and Morosi, [MM84, p. 27], [Fer94a, p. 4])

1. $I_k = \frac{1}{k}\mathrm{tr} N^k$ *satisfy* $N^* \mathrm{d} I_k = \mathrm{d} I_{k+1}$ *and* $I_1, I_2, \ldots$ *are in involution with respect to* P *and* Q;
2. $\lambda_1, \ldots, \lambda_n$ *are in involution with respect to* P *and* Q;
3. *If* $H, F \in C^\infty(M, \mathbb{R})$ *are such that* $X := P\mathrm{d}H = Q\mathrm{d}F$ *then* $X \in \mathfrak{X}_1$ *and* $\lambda_1, \ldots, \lambda_n$ *and* $I_1, I_2, \ldots$ *are constant along* X.

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Definition (hypersurface of translation, [Fer94a, p. 9])

Let $S \subset \mathbb{R}^{n+1}$ be a hypersurface. S is a **hypersurface of translation** if there exists a diffeomorphism $x = (x^1, \dots, x^{n+1})\colon \mathbb{R}^n \longrightarrow S \subset \mathbb{R}^{n+1}$, $t \longmapsto x(t)$, such that x is of the form

$$x^l(t^1, \dots, t^n) = a_1^l(t^1) + \cdots + a_n^l(t^n), \quad l = 1, \dots, n+1.$$

From now until the end of section 5, we use the following assumptions, which correspond to the situation we would be in if we applied the Arnold-Liouville theorem to some Hamiltonian system and considered a neighbourhood of an invariant torus.

Assumptions (1/2)

(Phase space) $U \subset \mathbb{R}^n$ is an open, nonempty, connected, simply connected neighbourhood of 0. $M := U \times T^n$, with coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ (the coordinates x on U are action coordinates and y on T^n are angle coordinates).

(Symplectic structure) $\omega = \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}y^i$.$\omega$ can be seen as a map $\omega \colon \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ and ω has an associated Poisson structure, which as a map $\Omega^1(M) \longrightarrow \mathfrak{X}(M)$ is given by $P := \omega^{-1}$.As a bivector, $P = -\sum_{i=1}^n \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}$. As a bracket, the Poisson structure is given by $\{f, g\} = P(\mathrm{d}f, \mathrm{d}g) = -\omega(X_f, X_g)$.The Hamiltonian vector field is given by $X_H^P = P\mathrm{d}H$ and $\mathrm{d}H = \omega(X_H^P, \cdot)$. With these conventions, $H \longmapsto X_H^P$ is a Lie algebra homomorphism.

Assumptions (2/2)

(Hamiltonian + completely integrable) $H\colon U \longrightarrow \mathbb{R}$ is the Hamiltonian function we consider on $U \times T^n$, which is independent of y: $H = H(x^1, \ldots, x^n)$.The system (M, ω, H) is completely integrable, i.e. there exist $F_1, \ldots, F_n \in C^\infty(U, \mathbb{R}) \subset C^\infty(M, \mathbb{R})$ which are independent, in involution and first integrals.

(Nondegeneracy) $\det\left(\frac{\partial^2 H}{\partial x^i \partial x^j}\right) \neq 0$ on an open dense set and the set $\{x \in U \mid \{x\} \times T^n \text{ is a non-resonant torus}\}$ is dense in U. Note: this implies that any first integral only depends on the action variables.

For $O \subset U$ open and $V \subset T^n$ open nonempty define conditions

$A(O, V) :=$ there exists Q a Poisson structure on $O \times V \subset M$ and there exists a function $F \colon O \times V \longrightarrow \mathbb{R}$ such that P, Q is bi-Poisson, $X_H^P = X_F^Q$ and $N = QP^{-1}$ has nice eigenvalues.

$B(O) :=$ The graph of H, $\mathrm{Graph}(H) \subset O \times \mathbb{R}$, is a hypersurface of translation.

Theorem (Fernandes, [Fer94a, p. 10])

1. *$A(U, T^n) \Longrightarrow \forall p \in U \colon \exists U'$ a neighbourhood of p in $U \colon B(U')$;*
2. *$B(U) \Longrightarrow \exists U' \subset U$ open :*
   $\exists V \subset T^n$ open and nonempty: $A(U', V)$.

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- We assume that there exists Q a Poisson structure on $U \times T^n \subset M$ and there exists a function $F\colon U \times T^n \longrightarrow \mathbb{R}$ such that P, Q is bi-Poisson, $X_H^P = X_F^Q$ and $N = QP^{-1}$ has nice eigenvalues $\lambda^1, \dots, \lambda^n$.
- We want to show that Graph(H) is a hypersurface of translation.
- We denote the corresponding eigenspaces by $E^1, \dots, E^n$, which are all 2 dimensional.
- By the nondegeneracy assumption, $\lambda^1, \dots, \lambda^n$ don't depend on y, or equivalently they are functions $\lambda^1, \dots, \lambda^n\colon U \longrightarrow \mathbb{R}$.
- Consider the map $\Lambda = (\lambda^1, \dots, \lambda^n)\colon U \longrightarrow \mathbb{R}^n$. Since $\lambda^1, \dots, \lambda^n$ are independent, $\Lambda\colon U \longrightarrow \mathbb{R}^n$ is a local diffeomorphism.
- Then, for $p \in U$ there exists $U' \subset U$ an open neighbourhood of p and $V \subset \mathbb{R}^n$ open such that $\Lambda\colon U' \longrightarrow V$ is a diffeomorphism. Denote the variables in V by $q^1, \dots, q^n$.

Definition (distributions, [Fer94a, p. 5, 6])

Define $\mathbf{n} = \{1, \ldots, n\}$ and $\hat{i} = \mathbf{n} \setminus \{i\}$. For $I \subset \mathbf{n}$, define

$$E^I := \bigoplus_{i \in I} E^i, \qquad (2|I| \text{ dimensional distribution on } U \times T^n)$$

$$D^I := \bigcap_{i \in \mathbf{n} \setminus I} \ker \mathrm{d}\lambda^i, \quad (|I| \text{ dimensional distribution on } U).$$

Proposition (properties of E^I, [Fer94a, p. 5-7])

For $I \subset \mathbf{n}$,

1. *E^I is integrable, with foliation which we denote by $\{L^I_\alpha\}_{\alpha \in J^I}$;*
2. *For all $j \in \mathbf{n} \setminus I$, λ^j is constant along the leaves of E^I;*
3. *$(E^I)^\omega = E^{\mathbf{n} \setminus I}$;*
4. *$L_{X_H^P}$ maps $\mathfrak{X}(E^I)$ to $\mathfrak{X}(E^I)$.*

Proposition (properties of D^I, [Fer94a, p. 7])

For $I \subset \mathbf{n}$,

1. *D^I is integrable, with foliation which we denote by $\{S^I_\beta\}_{\beta \in K^I}$;*
2. *For all $j \in \mathbf{n} \setminus I$, λ^j is constant along the leaves of D^I;*
3. *$D^I \oplus D^{\mathbf{n} \setminus I} = TU$.*

Proposition (relation between E^I and D^I, [Fer94a, p. 7])

For $I \subset \mathbf{n}$,

1. $\forall p \in M \colon \mathrm{D}\pi(p) E^I_p = D^I_{\pi(p)}$;
2. $\forall p \in M \colon \pi(L^I_p) = S^I_{\pi(p)}$.

Theorem (graph of H is hypersurface of translation, [Fer94a, p. 7])

Consider $H\colon V \longrightarrow \mathbb{R}$, $H = H(q)$ *and* $\Lambda^{-1}\colon V \longrightarrow U$, $\Lambda^{-1}(q) = (x^1(q), \ldots, x^n(q))$. *Then,*

$$\frac{\partial^2 H}{\partial q^i \partial q^j} = 0 \quad \textit{if } i \neq j,$$
$$\frac{\partial^2 x^k}{\partial q^i \partial q^j} = 0 \quad \textit{if } i \neq j.$$

In particular, $\mathrm{Graph}(H) \subset U' \times \mathbb{R}$ *is a hypersurface of translation.*

Step 1: There exist $O, W \subset T^n$ open and nonempty such that

$$\begin{aligned}\Gamma\colon U' \times O &\longrightarrow W \\ (x, y) &\longmapsto (\mathrm{D}\Lambda(x)^T)^{-1} y\end{aligned}$$

is well defined and $\Lambda \times \Gamma\colon U' \times O \longrightarrow V \times W$ is a symplectomorphism. Denote by $p^1, \ldots, p^n$ the coordinates on W, so that $q^1, \ldots, q^n, p^1, \ldots, p^n$ are symplectic coordinates on $V \times W$.

Step 2: Define $\overline{E}^I := (\Lambda \times \Gamma)_* E^I$, $\overline{N} := (\Lambda \times \Gamma)_* N$, $\overline{\lambda}^i := (\Lambda \times \Gamma)_* \lambda^i$ and $\overline{\Lambda} := (\Lambda \times \Gamma)_* \Lambda = \mathrm{id}_{\mathbb{R}^n} \colon V \times W \longrightarrow V \hookrightarrow \mathbb{R}^n$. Then,

1. There exists $a_{ij} \colon V \times W \longrightarrow \mathbb{R}$ (for $i, j = 1, \ldots, n$) such that for all $i = 1, \ldots, n$ we have that $a_{ii} = 0$ and

$$\overline{E}^i = \operatorname{span}\Big\{ X_{q^i}, X_{p^i} + \sum_{j=1}^{n} a_{ij} X_{q^j} \Big\}.$$

2. Denote also by $\overline{\Lambda}$ the map $\overline{\Lambda} := \operatorname{diag}(\overline{\lambda}^1, \ldots, \overline{\lambda}^n) \colon V \times W \longrightarrow \mathbb{R}^{n \times n}$. Define $\overline{B} \colon V \times W \longrightarrow \mathbb{R}^{n \times n}$ by $\overline{B}_{ij} := (\lambda_j - \lambda_i) a_{ji}$. Then,

$$\overline{N} = \begin{bmatrix} \overline{\Lambda} & 0 \\ \overline{B} & \overline{\Lambda} \end{bmatrix}.$$

Step 3: If f, g are functions on $U' \times O$ such that $P\mathrm{d}f = Q\mathrm{d}g$ and f is independent of y, then the corresponding functions $\overline{f}, \overline{g}\colon V \times W \longrightarrow \mathbb{R}$ satisfy

$$\frac{\partial^2 \overline{f}}{\partial q^i \partial q^j} = 0, \quad \frac{\partial^2 \overline{g}}{\partial q^i \partial q^j} = 0, \quad \text{for } i \neq j.$$

Step 4: $\frac{\partial^2 \overline{H}}{\partial q^i \partial q^j} = 0$, if $i \neq j$.

Step 5: $\frac{\partial^2 \overline{x}^k}{\partial q^i \partial q^j} = 0$, if $i \neq j$.

Proof of step 5

We start by proving that the entries of N do not depend on $p^1, \ldots, p^n$ and then we prove the following chain of implications:

$$
\begin{aligned}
&\text{Entries of } N \text{ do not depend on } p^1, \ldots, p^n \\
&\quad \Longrightarrow \text{Entries of } N \text{ do not depend on } y^1, \ldots, y^n \\
&\quad \Longrightarrow L_{X_{x^i}} N = L_{\partial_{y^i}} N = 0 \\
&\quad \Longrightarrow X_{x^i} \text{ is (locally) bi-Hamiltonian on open dense subset of } U' \\
&\quad \Longrightarrow \frac{\partial^2 x^k}{\partial q^i \partial q^j} = 0 \text{ for } i \neq j \text{, on open dense subset of } U' \\
&\quad \Longrightarrow \frac{\partial^2 x^k}{\partial q^i \partial q^j} = 0 \text{ for } i \neq j.
\end{aligned}
$$

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Step 1: using the fact that Graph(H) *is a hypersurface of translation.*
Since Graph(H) is a hypersurface of translation, there exists a diffeomorphism $\phi\colon O \longrightarrow U$ such that if we denote the variables in O by $t^1, \ldots, t^n$, then ϕ is of the form

$$\begin{aligned} x^1(t^1, \ldots, t^n) &= X^1_1(t^1) + \cdots + X^1_n(t^n) \\ &\vdots \\ x^n(t^1, \ldots, t^n) &= X^n_1(t^1) + \cdots + X^n_n(t^n) \\ H(x(t)) &= H_1(t^1) + \cdots + H_n(t^n). \end{aligned}$$

Step 2: define new symplectic coordinates $q^1, \dots, q^n, p^1, \dots, p^n$ such that $H(q) = q^1 + \cdots + q^n$. (Possibly at the cost of shrinking the domain).

Step 3: writing Q and N on the new coordinates. On $W \times Z$,

- Define $Q(q,p) = \sum_{i=1}^n q^i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i}$. Then Q is Poisson.
- (P, Q) is bi-Poisson.
- $N = \sum_{i=1}^n q^i \Big(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p^i} \otimes \mathrm{d}p^i \Big)$ and N has nice eigenvalues.
- $L_{X_H^P} N = 0$ and X_H^P is bi-Hamiltonian.

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- **Example: symmetric top rotating freely about a fixed point** ([Fer94a, p. 13, 14]). We will see that the Hamiltonian has a graph which is a hypersurface of translation. We will use Fernandes' theorem to build a new Poisson structure Q on the phase space.
- **Phase space (Poisson):** $\mathfrak{so}(3)$ acts on $\mathbb{R}^3$ by multiplication on the left. We can use this action to define the semidirect sum $\mathfrak{e}(3) := \mathfrak{so}(3) \oplus \mathbb{R}^3$, which is a Lie algebra ([Ost13, p. 237-239]). It's dual, $\mathfrak{e}(3)^*$ is a Poisson manifold and is going to be our Phase space ([BRS89, p. 326]). $\mathfrak{e}(3)^*$ has coordinates $(M_1, M_2, M_3, p_1, p_2, p_3)$ and the Poisson bracket is given by

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0.$$

- **Hamiltonian:**

$$H = \frac{1}{2I_1}(M_1^2 + M_2^2) + \frac{1}{2I_3} M_3^2.$$

- **Casimir functions:** Define $C_1 = p_1^2 + p_2^3 + p_3^2$ and $C_2 = p_1 M_1 + p_2 M_3 + p_3 M_3$. Then, C_1, C_2 are Casimir functions: $C_1, C_2 \in \ker(F \longmapsto X_F)$.
- **Phase space (symplectic):** Restrict to $L = \{C_1 = 1, C_2 = 0\}$. L is a leaf of the Kirillov foliation, so L is a symplectic manifold. (L, ω, H) is completely integrable with first integrals H, M_3.
- **Coordinates on L:** Define new coordinates $(\theta, \varphi, p_\theta, p_\varphi)$ on L by the equations

$$\begin{aligned} p_1 &= \cos\theta\cos\varphi & M_1 &= p_\varphi \tan\theta\cos\varphi - p_\theta \sin\varphi \\ p_2 &= \cos\theta\sin\varphi & M_2 &= p_\varphi \tan\theta\sin\varphi + p_\theta \cos\varphi \\ p_3 &= \sin\theta & M_3 &= p_\varphi. \end{aligned}$$

$$\begin{aligned} \omega &= \mathrm{d}\theta \wedge \mathrm{d}p_\theta + \mathrm{d}\varphi \wedge \mathrm{d}p_\varphi \\ H &= p_\varphi^2\left(\frac{1}{2I_1}\tan^2\theta + \frac{1}{2I_3}\right) + \frac{1}{2I_1}p_\theta^2. \end{aligned}$$

- **Invariant torus:** Choose $h, m \in \mathbb{R}$ such that $h/m^2 > 1/(2I_3)$ and define $T = \{H = h, M_3 = m\}$.
- **Action variables:** the action variables and H are given by

$$
\begin{aligned}
s_1 &= m \\
s_2 &= \sqrt{2hI_1 - m^2 \frac{I_1 - I_3}{I_3}} - m \\
H &= \frac{I_1 - I_3}{2I_1 I_3} s_1^2 + \frac{1}{2I_1}(s_1 + s_2)^2.
\end{aligned}
$$

- **Graph of H is a hypersurface of translation:** by using the parametrization $\mathbb{R}^2 \longrightarrow \mathrm{Graph}(H)$ given by $(t_1, t_2) \longmapsto s_1(t), s_2(t), s_3(t)$,

$$
\begin{aligned}
s_1 &= t_1 \\
s_2 &= t_2 - t_1 \\
s_3 &= \frac{I_1 - I_3}{2I_1 I_3} t_1^2 + \frac{1}{2I_1} t_2^2.
\end{aligned}
$$

- **New Poisson structure:** So, by Fernandes' theorem, we can conclude that there exists a new Poisson structure Q. Following the proof of this theorem, we should be able to conclude that it's given by

$$
\begin{aligned}
Q = {} & \frac{I_1 - I_3}{2 I_1 I_3} s_1^2 \frac{\partial}{\partial s_1} \wedge \frac{\partial}{\partial \theta_1} \\
& + \left(\frac{I_1 - I_3}{2 I_1 I_3} s_1^2 - \frac{1}{2 I_1} (s_1 + s_2)^2 \right) \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial s_2} \\
& + \frac{1}{2 I_1} (s_1 + s_2)^2 \frac{\partial}{\partial s_2} \wedge \frac{\partial}{\partial \theta_2}.
\end{aligned}
$$

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Thank you for listening!