Symplectic capacities

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Definition (symplectic manifold)

A **symplectic manifold** is a pair (M, ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate. An **exact symplectic manifold** is a pair (M, θ) such that $(M, \mathrm{d}\theta)$ is a symplectic manifold.

Definition (symplectic embedding)

Let (M, ω_M), (N, ω_N) be symplectic manifolds. A **symplectic embedding** from M to N is an embedding $f\colon M \longrightarrow N$ such that $f^*\omega_N = \omega_M$. If (M, θ_M), (N, θ_N) are exact symplectic manifolds, an **exact symplectic embedding** from M to N is an embedding $f\colon M \longrightarrow N$ such that $f^*\theta_N = \theta_M$;

Example (symplectic manifolds)

1. Consider $\mathbb{R}^{2n}$ with coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$. Define $\theta_0 = \sum_{j=1}^n x^j \mathrm{d}y^j$. Then, $\omega_0 = \mathrm{d}\theta_0 = \sum_{j=1}^n \mathrm{d}x^j \wedge \mathrm{d}y^j$ is symplectic and $(\mathbb{R}^{2n}, \theta_0)$ is an exact symplectic manifold.
2. Let M be a manifold and consider it's cotangent bundle $\pi\colon T^*M \longrightarrow M$. Define a form $\theta \in \Omega^1(T^*M)$, called the **canonical symplectic potential**, or **Liouville form**, by

$$\begin{tikzcd} T_{(p,\alpha)}(T^*M) \arrow[d, "\mathrm{d}\pi(p,\alpha)"'] \arrow[dr, "\theta_{(p,\alpha)}"] & \\ T_pM \arrow[r, "\alpha"'] & \mathbb{R}. \end{tikzcd}$$

   Then, (T^*M, θ) is an exact symplectic manifold.

Theorem (Darboux' theorem and implications)

Let (M, ω) be a symplectic manifold. Then,

1. *For every $p \in M$, there exists a coordinate neighborhood $(U, q_1, \dots, q_n, p_1, \dots, p_n)$ of p such that*

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

2. *M is even dimensional.*
3. *ω^n is a volume form on M, and M is orientable.*

Definition (Lagrangian submanifold)

Let (M, ω) be a $2n$-dimensional symplectic manifold and $\iota \colon L \longrightarrow M$ be an embedded submanifold. L is **Lagrangian** if $\dim L = n$ and $\iota^* \omega = 0$.

Remark (notation)

In this section, $\mathbf{SMan}(n)$ denotes the category whose objects are symplectic manifolds of dimension $2n$ and whose morphisms are symplectic embeddings.

Example (subsets of $\mathbb{C}^n$)

The ball, ellipsoid and cylinder are subsets of $\mathbb{C}^n \cong \mathbb{R}^{2n}$:

$$\begin{aligned} B(r) &= \{z \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 \leq r^2\}, \\ E(r_1, \ldots, r_n) &= \left\{z \in \mathbb{C}^n \;\middle|\; \frac{|z_1|^2}{r_1^2} + \cdots + \frac{|z_1|^2}{r_n^2} \leq 1\right\}, \\ Z(r) &= \{z \in \mathbb{C}^n \mid |z_1|^2 \leq r^2\}. \end{aligned}$$

So, all three of them are exact symplectic manifolds.

Definition (symplectic capacity)

A **symplectic capacity** is a function $c\colon \mathbf{SMan}(n) \longrightarrow [0,\infty]$ such that

(Monotonicity) If there exists a symplectic embedding $\varphi\colon (M,\omega_M) \longrightarrow (N,\omega_N)$, then $c(M,\omega_M) \leq c(N,\omega_N)$;

(Conformality) For all $\alpha \in \mathbb{R} \setminus \{0\}$ and for all (M,ω) a $2n$-dimensional symplectic manifold we have that $c(M,\alpha\omega) = |\alpha| c(M,\omega)$;

(Nontriviality) $0 < c(B(1),\omega_0)$ and $c(Z(1),\omega_0) < +\infty$.

Definition (normalization)

A symplectic capacity c satisfies the **normalization** axiom if $0 < c(B(1),\omega_0) = \pi = c(Z(1),\omega_0) < +\infty$.

Theorem (Gromov's nonsqueezing)

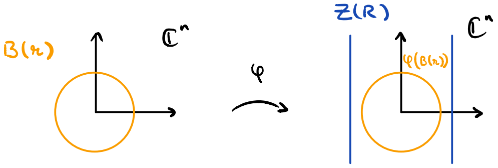
There exists a symplectic embedding $\varphi \colon B(r) \longrightarrow Z(R)$ if and only if $r \leq R$.

Definition (Gromov width)

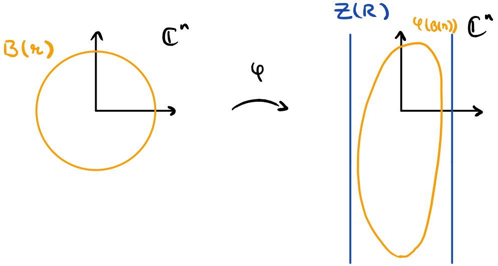
The **Gromov width** is the function $c_{\mathrm{Gr}} \colon \mathbf{SMan}(n) \longrightarrow [0, \infty]$ given by

$$c_{\mathrm{Gr}}(M, \omega) = \sup\{\pi r^2 \mid \exists \text{ symplectic embedding } \varphi \colon B(r) \longrightarrow M\}.$$

$r \leq R$: Can happen that



$r > R$: Can't happen that



Proposition (Gromov width is the smallest capacity)

Assume that c_{Gr} is a normalized symplectic capacity and that c is a symplectic capacity. Then, for every symplectic manifold (M, ω) we have

$$c_{\mathrm{Gr}}(M, \omega) \leq \frac{\pi}{c(B(1))} c(M, \omega).$$

Proposition (Gromov's nonsqueezing $\Longleftrightarrow \exists$ normalized capacity)

The following are equivalent

1. *The Gromov width is a normalized symplectic capacity;*
2. *There exists a normalized symplectic capacity;*
3. *Gromov's nonsqueezing theorem is true, i.e. for $r, R > 0$ there exists a symplectic embedding $B(r) \hookrightarrow Z(R)$ if and only if $r \leq R$.*

Proof.

1 $\implies$ 2: Trivial.

2 $\implies$ 3: If $r \leq R$, then $\iota \colon B(r) \longrightarrow Z(R)$ is a symplectic embedding. Conversely,

$$\begin{aligned}
r^2\pi &= r^2 c(B(1)) && [c \text{ is normalized}] \\
&= c(B(r)) && [\text{capacity of rescaled set}] \\
&\leq c(Z(R)) && [\text{by assumption and monotonicity}] \\
&= R^2 c(Z(1)) && [\text{capacity of rescaled set}] \\
&= R^2\pi && [c \text{ is normalized}].
\end{aligned}$$

Proof (Cont.)

$3 \implies 1$: That c_{Gr} satisfies monotonicity and conformality is a simple, if a bit lengthy, proof. The proof of these properties being true does not depend on Gromov's nonsqueezing theorem being true. It remains to show normalization, i.e. that $c_{\mathrm{Gr}}(B(1)) = \pi$ and $c_{\mathrm{Gr}}(Z(1)) = \pi$. For this, define

$$\begin{aligned} \mathcal{S} &= \{\pi r^2 \mid r \leq 1\}, \\ \mathcal{S}_B &= \{\pi r^2 \mid \text{there exists a symplectic embedding } B(r) \hookrightarrow B(1)\}, \\ \mathcal{S}_Z &= \{\pi r^2 \mid \text{there exists a symplectic embedding } B(r) \hookrightarrow Z(1)\}. \end{aligned}$$

Then, $\mathcal{S} \subset \mathcal{S}_B \subset \mathcal{S}_Z \subset \mathcal{S}$ by Gromov's nonsqueezing theorem. Therefore, $c_{\mathrm{Gr}}(B(1)) = \sup \mathcal{S}_B = \sup \mathcal{S} = \pi$, and analogously $c_{\mathrm{Gr}}(Z(1)) = \sup \mathcal{S}_Z = \sup \mathcal{S} = \pi$. $\square$

Table: Examples of symplectic capacities. See the introduction of [GH18].

notation	name	domain	technique	reference
c_k, $k \in \mathbb{N}$	equivariant capacities	Liouville domains	S^1-equivariant symplectic homology	[GH18]
c_k^{EH}, $k \in \mathbb{N}$	Ekeland-Hofer capacities	compact star-shaped domains in $\mathbb{R}^{2n}$	calculus of variations for the symplectic action functional on $C^\infty(S^1, \mathbb{R}^{2n})$	[EH89]
c_k^{ECH}, $k \in \mathbb{N}_0$	ECH capacities	4-dimensional Liouville domains	embedded contact homology	[Hut10]
$c_\bigcirc$, c_{Gr}	embedding of ball, Gromov width	symplectic manifolds	embedding capacity	-
$c_\square$	embedding of cube	symplectic manifolds	embedding capacity	-
c_L	Lagrangian capacity	symplectic manifolds	-	[CM18]
c_0	-	symplectic manifolds	oscillation of admissible Hamiltonians	[HZ11]

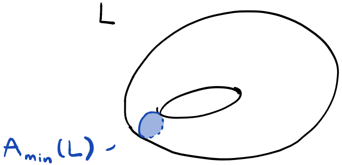
Remark ($\omega\colon \pi_2(M, L) \longrightarrow \mathbb{R}$)

Let (M, ω) be a symplectic manifold and $L \subset M$ be a Lagrangian submanifold. Then, ω can be seen as a map $\pi_2(M, L) \longrightarrow \mathbb{R}$ as follows. An element of $\pi_2(M, L)$ can be seen as an equivalence class of a map $\sigma\colon (D, S^1) \longrightarrow (M, L)$. Then, $\omega([\sigma]) = \int_D \sigma^* \omega$. To show that this is well defined, we need to use the fact that L is Lagrangian.

Definition (minimal symplectic area of a Lagrangian submanifold)

Let (X,ω) be a symplectic manifold. If L is a Lagrangian submanifold of X, then we define the **minimal symplectic area of** L, $A_{\min}(L)$, by

$$\begin{aligned} A_{\min}(L) &:= \inf\{\omega(\sigma) \mid \sigma \in \pi_2(X,L),\ \omega(\sigma) > 0\} \\ &= \inf\left\{ \int_D u^*\omega \;\middle|\; u\colon (D,\partial D) \longrightarrow (X,L),\ \int_D u^*\omega > 0 \right\} \\ &\in [0,\infty]. \end{aligned}$$



Lemma ($A_{\min}$ with exact ambient symplectic manifold)

Let (M, λ) be an exact symplectic manifold and $L \subset M$ a Lagrangian submanifold. If $\pi_1(M) = \{0\}$, then

$$A_{\min}(L) = \inf \{\lambda(\rho) \mid \rho \in \pi_1(L),\ \lambda(\rho) > 0\}.$$

Proof.

The following diagram commutes

$$\begin{tikzcd}
\pi_2(L) \arrow[r] \arrow[d, "0"'] & \pi_2(M) \arrow[r] \arrow[d, "\omega"] & \pi_2(M, L) \arrow[r, two heads, "d"] \arrow[d, "\omega"] & \pi_1(L) \arrow[r, "0"] \arrow[d, "\lambda"] & \pi_1(M) \arrow[d, "\lambda"] \\
\mathbb{R} \arrow[r, hook, two heads, "\mathrm{id}"'] & \mathbb{R} \arrow[r, hook, two heads, "\mathrm{id}"'] & \mathbb{R} \arrow[r, hook, two heads, "\mathrm{id}"'] & \mathbb{R} \arrow[r, hook, two heads, "\mathrm{id}"'] & \mathbb{R}
\end{tikzcd},$$

where $d([\sigma]) = [\sigma|_{S^1}]$ and the top row is exact. $\square$

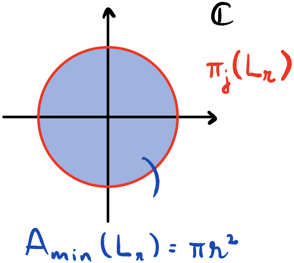
Lemma ($A_{\min}$ of a Lagrangian torus in $\mathbb{C}^n$)

For $r > 0$, let $L_r = \{z \in \mathbb{C}^n \mid |z_1|^2 = \cdots = |z_n|^2 = r^2\}$. Then, L_r is a Lagrangian submanifold of $\mathbb{C}^n$ and $A_{\min}(L_r) = \pi r^2$.

Proof.

Use the previous lemma. □

$\pi_j : \mathbb{C}^n \to \mathbb{C}$



Definition (Lagrangian capacity)

We define the **Lagrangian capacity of** (X, ω), $c_L(X, \omega)$, by

$$c_L(X, \omega) \coloneqq \sup\{A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus}\} \in [0, \infty].$$

Proposition (Properties of the Lagrangian capacity)

The Lagrangian capacity c_L satisfies:

(Monotonicity) If $\iota \colon (X, \omega) \longrightarrow (X', \omega')$ is a symplectic embedding s.t. $\pi_2(X', \iota(X)) = 0$, then $c_L(X, \omega) \leq c_L(X', \omega')$.

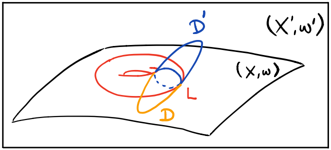
(Conformality) For all $\alpha \in \mathbb{R} \setminus \{0\}$, $c_L(X, \alpha\omega) = |\alpha| c_L(X, \omega)$.

Proof (of Monotonicity).

It suffices to assume that $L \subset X$ is an embedded Lagrangian torus and to prove that $A_{\min}(L, X) = A_{\min}(\iota(L), X')$.

$(\geq)$: Easy, doesn't depend on $\pi_2(X', \iota(X)) = \{0\}$.

$(\leq)$: It suffices to assume that D' is a disk as in the definition of $A_{\mathrm{min}}(\iota(L), X')$ and to prove that there exists a disk D as in the definition of $A_{\mathrm{min}}(L, X)$ such that $\int_D \omega = \int_{D'} \omega$. Since $\pi_2(X', \iota(X)) = \{0\}$, there exists a disk D as in the definition of $A_{\mathrm{min}}(L, X)$ such that D and D' have the same boundary and the sphere obtained by gluing D and D' has homotopy class 0. The disk D is as desired. □



$D' \in \pi_2(X', \iota(L))$

$D \in \pi_2(X, L)$

Proposition (Lagrangian capacity of the ball)

$c_L(B(1)) = \pi/n$.

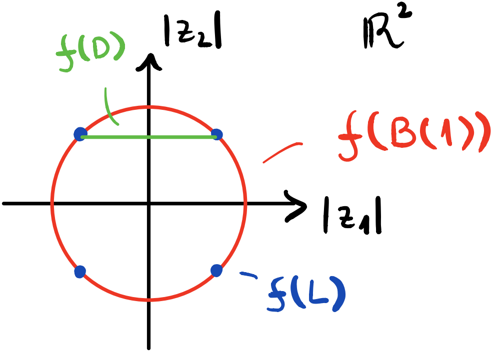
Proof.

$(\geq)$: It suffices to show that there exists $L \subset B(1)$ an embedded Lagrangian torus such that $A_{\min}(L) = \pi/n$. Define

$$L = \{z \in \mathbb{C}^n \mid |z_1|^2 = \cdots = |z_n|^2 = 1/n\} \subset B(1).$$

Then, by a previous lemma, we have that $A_{\min}(L) = \pi/n$.
$(\leq)$: This is hard, depends on the main theorem of [CM18] (this theorem says that there are disks with boundary on a Lagrangian of small area, and it's proof uses holomorphic curve techniques). $\square$

Drawing illustrating the proof of $c_L(B(1)) \geq \pi/n$: Define $f: \mathbb{C}^n \longrightarrow \mathbb{R}^n$ via $f(z_1, \ldots, z_n) = (|z_1|, \ldots, |z_n|)$. Then, for the case $n = 2$:



Proposition (Lagrangian capacity of the cylinder)

$c_L(Z(1)) = \pi$.

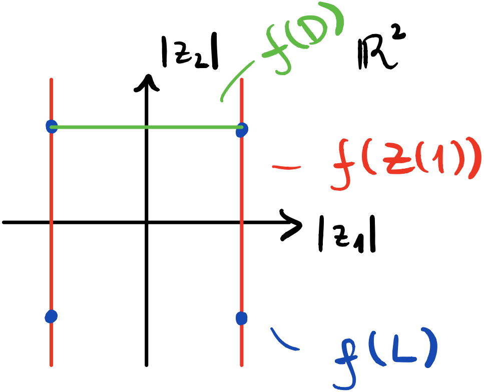
Proof.

$(\geq)$: It suffices to show that there exists $L \subset Z(1)$ an embedded Lagrangian torus such that $A_{\min}(L) = \pi$. Define

$$L = \{z \in \mathbb{C}^n \mid |z_1|^2 = \cdots = |z_n|^2 = 1\} \subset Z(1).$$

Then, by a previous lemma, we have that $A_{\min}(L) = \pi$.
$(\leq)$: Again, this is hard. Depends on the concepts of Hofer norm, Hofer energy, displacement energy, and a result of Chekanov comparing minimal area of a Lagrangian and displacement energy. See [CM18, HZ11, Che98]. □

Drawing illustrating the proof of $c_L(Z(1)) \geq \pi$, for the case $n = 2$:



Corollary (Lagrangian capacity of the ellipsoid)

Let $E(r_1, \dots, r_n) \subset \mathbb{C}^n$ *be an ellipsoid and let* $r_k = \min\{r_1, \dots, r_n\}$*. Then,*

$$\frac{\pi}{n} r_k^2 \leq \pi \left(\frac{1}{r_1^2} + \cdots \frac{1}{r_n^2} \right)^{-1} \leq c_L(E(r_1, \dots, r_n)) \leq \pi r_k^2.$$

Proof.

Notice that $B(r_k) \subset E(r_1, \dots, r_n) \subset Z(r_k)$. Define

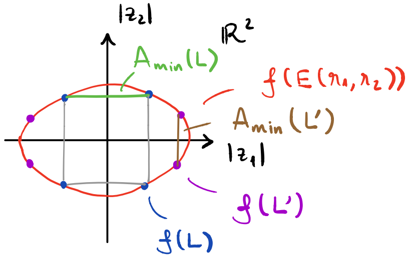
$$r^2 = \left(\frac{1}{r_1^2} + \cdots \frac{1}{r_n^2} \right)^{-1}$$

and $L = \{z \in \mathbb{C}^n \mid |z_1|^2 = \cdots = |z_n|^2 = r^2\} \subset E(r_1, \dots, r_n)$. Then, by a previous lemma, we have that $A_{\min}(L) = \pi \left(\frac{1}{r_1^2} + \cdots \frac{1}{r_n^2} \right)^{-1}$.

Proof (Cont.)

$$
\begin{aligned}
\pi r_k^2 &= c_L(Z(r_k)) && \text{[by a rescaling property of capacities]} \\
&\geq c_L(E(r_1, \ldots, r_n)) && [E(r_1, \ldots, r_n) \subset Z(r_k)] \\
&\geq \pi \left(\frac{1}{r_1^2} + \cdots \frac{1}{r_n^2} \right)^{-1} && \text{[by the computation with } L \text{ above]} \\
&\geq \frac{\pi}{n} r_k^2 && \text{[because } r_k = \min\{r_1, \ldots, r_k\}\text{].}
\end{aligned}
$$

□



Conjecture (Lagrangian capacity of ellipsoid, [CM18])

Let $E(r_1, \ldots, r_n) \subset \mathbb{C}^n$ *be an ellipsoid. Then,*

$$c_L(E(r_1, \ldots, r_n)) = \pi \left(\frac{1}{r_1^2} + \cdots \frac{1}{r_n^2} \right)^{-1}.$$

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Thank you for listening!