Symplectic capacities part 2

A comparison of two symplectic capacities

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Conjecture (comparison between c_L and c_k)

If (X, λ) is a Liouville domain which satisfies $\pi_1(X) = \{0\}$ and $c_1(TX)|_{\pi_2(X)} = 0$, then

$$c_L(X, \lambda) \leq \inf_{k \in \mathbb{N}} \frac{c_k(X, \lambda)}{k}.$$

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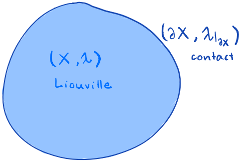
Definition (symplectic manifold)

A **symplectic manifold** is a pair (M, ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate. An **exact symplectic manifold** is a pair (M, θ) such that $(M, \mathrm{d}\theta)$ is a symplectic manifold.

Definition (Liouville domain)

A **Liouville domain** is a pair (X, λ), where X is a compact, connected smooth manifold with boundary ∂X and $\lambda \in \Omega^1(X)$ is such that $\mathrm{d}\lambda \in \Omega^2(X)$ is symplectic, $\lambda|_{\partial X}$ is contact and the orientations on ∂X coming from $(X, \mathrm{d}\lambda)$ and coming from $\lambda|_{\partial X}$ are equal.



Definition (types of morphisms for symplectic manifolds)

Let (X, ω_X), (Y, ω_Y) be symplectic manifolds (possibly with boundary and corners) and $\varphi\colon X \longrightarrow Y$ be an embedding. Then, φ is **symplectic** if $\varphi^* \omega_Y = \omega_X$. A **symplectomorphism** is a symplectic embedding which is a diffeomorphism. We say that φ is **strict** if $\varphi(X) \subset \operatorname{int} Y$. If (X, λ_X), (Y, λ_Y) are exact symplectic, then we say that

1. φ is **symplectic** if $\varphi^* \lambda_Y - \lambda_X$ is closed (this is equivalent to the previous definition);
2. φ is **generalized Liouville** if $\varphi^* \lambda_Y - \lambda_X$ is closed and $\varphi^* \lambda_Y - \lambda_X$ is exact on ∂X;
3. φ is **exact** if $\varphi^* \lambda_Y - \lambda_X$ is exact;
4. φ is **Liouville** if $\varphi^* \lambda_Y - \lambda_X = 0$.

Definition (categories of symplectic manifolds)

- Define a category $\mathbf{S}$ whose objects are symplectic manifolds (possibly with boundary) and whose morphisms are 0-codimensional symplectic embeddings.
- Define a category $\mathbf{L}$ as follows. An object of $\mathbf{L}$ is a nondegenerate Liouville domain (X, λ) such that $\pi_1(X) = \{0\}$, $c_1(TX)|_{\pi_2(X)} = 0$. Morphisms in $\mathbf{L}$ are either identities or 0-codimensional strict exact symplectic embeddings $\phi\colon (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$.

Definition (symplectic capacity)

A **domain for a symplectic capacity** is a subcategory **D** of **S** such that $(M, \omega) \in \mathbf{D}$ implies $(M, \alpha\omega) \in \mathbf{D}$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. A **symplectic capacity** is a map $c \colon \mathbf{D} \longrightarrow [0, +\infty]$, where **D** is a domain for a symplectic capacity, such that

(Monotonicity) c is a functor, i.e. if $(M, \omega_M) \longrightarrow (N, \omega_N)$ is a morphism in **D** then $c(M, \omega_M) \leq c(N, \omega_N)$.

(Conformality) For every $\alpha \in \mathbb{R}$ and $(M, \omega) \in \mathbf{D}$ we have that $c(M, \alpha\omega) = |\alpha| c(M, \omega)$.

If $B(1), Z(1) \in \mathbf{D}$, then c is **nontrivial** or **normalized** if it satisfies (respectively):

(Nontriviality) $0 < c(B(1)) < c(Z(1)) < +\infty$.

(Normalization) $0 < c(B(1)) = 1 = c(Z(1)) < +\infty$.

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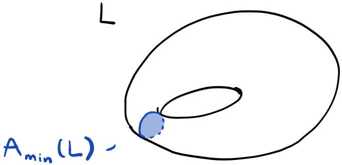
Definition (minimal symplectic area of a Lagrangian submanifold)

Let (X, ω) be a symplectic manifold. If L is a Lagrangian submanifold of X, then we define the **minimal symplectic area of** L, $A_{\min}(L)$, by

$$\begin{aligned} A_{\min}(L) &\coloneqq \inf \{\omega(\sigma) \mid \sigma \in \pi_2(X, L),\ \omega(\sigma) > 0\} \\ &= \inf \left\{ \int_D u^*\omega \;\middle|\; u\colon (D, \partial D) \longrightarrow (X, L),\ \int_D u^*\omega > 0 \right\} \\ &\in [0, \infty]. \end{aligned}$$



Definition (Lagrangian capacity)

We define the **Lagrangian capacity of** (X, ω), $c_L(X, \omega)$, by

$$c_L(X, \omega) \coloneqq \sup\{A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus}\} \\ \in [0, \infty].$$

Proposition (Properties of the Lagrangian capacity)

The Lagrangian capacity c_L satisfies:

(Monotonicity) If $\iota \colon (X, \omega) \longrightarrow (X', \omega')$ is a symplectic embedding s.t. $\pi_2(X', \iota(X)) = 0$, then $c_L(X, \omega) \leq c_L(X', \omega')$.

(Conformality) For all $\alpha \in \mathbb{R} \backslash \{0\}$, $c_L(X, \alpha\omega) = |\alpha| c_L(X, \omega)$.

Proposition (Lagrangian capacity of the ball, [CM18])

$c_L(B(1)) = 1/n$.

Proposition (Lagrangian capacity of the cylinder, [CM18])

$c_L(Z(1)) = 1$.

Conjecture (Lagrangian capacity of ellipsoid, [CM18])

Let $E(a_1, \ldots, a_n) \subset \mathbb{C}^n$ be an ellipsoid. Then,

$$c_L(E(a_1, \ldots, a_n)) = \left(\frac{1}{a_1} + \cdots \frac{1}{a_n} \right)^{-1}.$$

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Let (X, λ) be a nondegenerate Liouville domain. We will define $CH(X)$, the PS^1ESH of X.

Incomplete definition (completion)

The **completion** of (X, λ) is an exact symplectic manifold $(\hat{X}, \hat{\lambda})$ given as a set by $\hat{X} = X \cup_{\partial X} [0, +\infty) \times \partial X$.

Incomplete definition (some categories)

- $\mathbf{I}_X$ is a category whose objects are (H, J), where $H \colon S^1 \times \hat{X} \times S^{2M+1} \longrightarrow \mathbb{R}$ is an **admissible Hamiltonian** on $\hat{X}$ and $J \colon S^1 \times S^{2M+1} \longrightarrow \operatorname{End} T\hat{X}$ is an **admissible ACS** on $\hat{X}$.
- $\mathbf{C} = \mathbf{Comp}(\operatorname{Hom}(\mathbb{R}, {}_{\mathbb{Q}}\mathbf{Mod}))$ is the category of **complexes of filtered $\mathbb{Q}$-modules** (up to homotopy).
- $\mathbf{A} := \operatorname{Hom}(\mathbb{Z} \times \mathbb{R}, {}_{\mathbb{Q}}\mathbf{Mod}) =$ cat. of **graded, filtered modules**.
- **Homology** is a functor $H \colon \mathbf{C} \longrightarrow \mathbf{A}$.

Incomplete definition (PS^1EFH)

- **Positive S^1-equivariant Floer complex** is a covariant functor

$$\begin{aligned} FC^+_X \colon \qquad \mathbf{I}_X &\longrightarrow \mathbf{C} \\ (H^+, J^+) &\longmapsto FC^+_X(H^+, J^+) \\ \downarrow &\longmapsto \downarrow \phi^{-,+} \quad \text{(Floer continuation map)} \\ (H^-, J^-) &\longmapsto FC^+_X(H^-, J^-). \end{aligned}$$

  $FC^+_X(H, J)$ is generated by $[z, \gamma]$, where $z \in S^{2M+1}$ and γ is a 1-periodic orbit of H_z.

- **Positive S^1-equivariant Floer homology** is

$$FH^+_X = H \circ FC^+_X \colon \mathbf{I}_X \longrightarrow \mathbf{C} \longrightarrow \mathbf{A}.$$

Definition (PS^1ESH)

Positive S^1-equivariant symplectic homology is

$$CH(X, \lambda) = \operatorname{colim} FH^+_X \in \mathbf{A}.$$

Theorem (PS^1ESH)

Positive S^1-equivariant symplectic homology *is a contravariant functor $CH \colon \mathbf{L} \longrightarrow \mathbf{A}$.*

$$\begin{aligned} CH\colon \qquad \mathbf{L} &\longrightarrow \mathbf{A} \\ (X, \lambda_X) &\longmapsto CH(X, \lambda_X) \\ \varphi \downarrow &\longmapsto \uparrow \varphi_! \quad (\textit{Viterbo transfer map}) \\ (Y, \lambda_Y) &\longmapsto CH(Y, \lambda_Y). \end{aligned}$$

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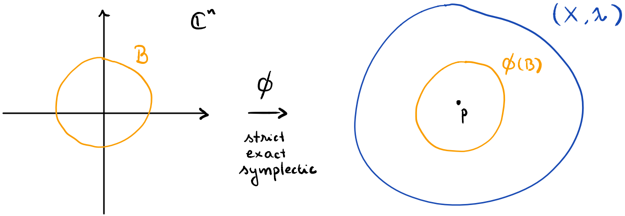
Definition (equivariant capacity)

The k**th equivariant capacity**, denoted c_k, is a symplectic capacity with domain $\mathbf{L}$ which is given, for $(X, \lambda) \in \mathbf{L}$, as follows. Choose $B \subset \mathbb{C}^n$ a nondegenerate star-shaped domain and $\phi \colon B \longrightarrow X$ a strict exact symplectic embedding. Then, $c_k(X)$ is the infimum over $a > 0$ such that the following map is surjective:

$$CH^a_{n+2k-1}(X) \xrightarrow{\iota^a} CH_{n+2k-1}(X) \xrightarrow{\phi_!} CH_{n+2k-1}(B) \ .$$



Remark (standard vs alternative definition of c_k)

Actually, the definition we gave of c_k is an alternative definition. The standard definition was given in [GH18].

- **Standard definition** ([GH18]): relies on additional properties of positive S^1-equivariant symplectic homology (which we did not mention), namely maps U and δ. This definition doesn't rely on choosing B.
- **Alternative definition**: it's possible to prove that the definition we gave and the one given in [GH18] are equivalent. Our definition doesn't depend on the maps U or δ, but depends on choosing B. We will only need the alternative definition to understand the proof of the conjecture.

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Step 1: suffices assume/prove

It suffices to assume that

- $(X, \lambda) \in \mathbf{L}$ (i.e. X is a nondegenerate Liouville domain with $\pi_1(X) = \{0\}$ and $c_1(TX)|_{\pi_2(X)} = 0$)
- $k \in \mathbb{N}$
- $L \subset \operatorname{int} X$ is an embedded Lagrangian torus
- $a > c_k(X, \lambda)$

and to prove that there exists $\sigma \in \pi_2(X, L)$ such that $0 < \omega(\sigma) \leq a/k$.

Proof of step 1.

By definition of c_L. □

Step 2: constructing a tubular neighbourhood

There exists g a Riemannian metric on L and there exists $W \subset \operatorname{int} X$ a closed set containing L such that there exists a symplectomorphism $\psi \colon W \longrightarrow D^* L$ and such that for every closed geodesic γ of L, if $l(\gamma) \leq a$ then γ is not contractible and nondegenerate and $0 \leq \mathrm{ind}_{\mathrm{M}}(\gamma) \leq n-1$.

Proof of step 2.

By the Lagrangian neighbourhood theorem plus a lemma from [CM18] which says that metrics of nonpositive sectional curvature (for example the flat metric on the torus) can be perturbed to have the desired property. □

Step 3: choosing a small ball inside W

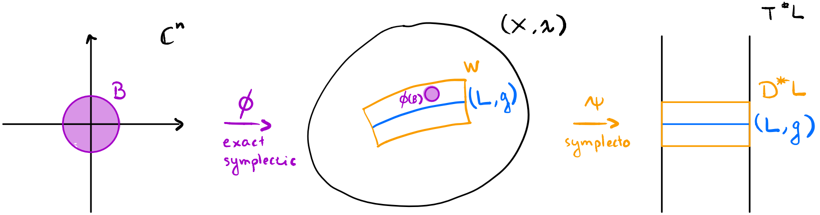
There exists $B \subset \mathbb{C}^n$ a nondegenerate Liouville domain and $\phi \colon B \longrightarrow X$ a strict exact symplectic embedding such that $\phi(B) \subset \operatorname{int} W$ and the following map is surjective:

$$CH^a_{n+2k-1}(X) \xrightarrow{\iota^a} CH_{n+2k-1}(X) \xrightarrow{\phi_!} CH_{n+2k-1}(B) \; .$$

Proof of step 3.

By definition of $c_k(X)$ and because $c_k(X) < a$. □

So, until now we have the following:



Next we will use the definition of PS^1ESH and the fact that the map

$$CH^{a}_{n+2k-1}(X) \xrightarrow{\iota^a} CH_{n+2k-1}(X) \xrightarrow{\phi_!} CH_{n+2k-1}(B)$$

is surjective/nonzero. From now on the proof will be less rigorous.

Step 4: choosing auxiliary data

Choose H an admissible Hamiltonian and J an admissible almost complex structure on $\hat{X}$. Do a construction from SFT (**symplectic field theory**) called **neck stretching**, which produces a sequence of admissible almost complex structures $(J_l)_{l\in\mathbb{N}}$ which are becoming singular on ∂W.

Step 5: applying the definition of CH

For every J_l there exist

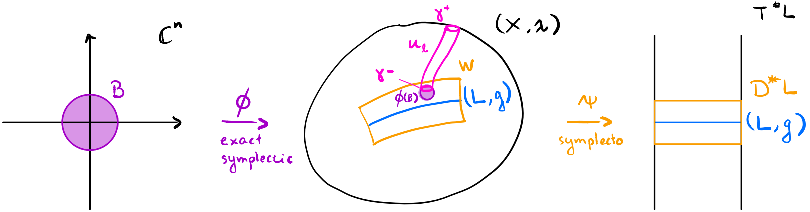
- $[z^+, \gamma^+]$ a generator of $FC^+_X(H, J_l)$ with γ^+ near ∂X;
- $[z^-, \gamma^-]$ a generator of $FC^+_X(H, J_l)$ with γ^- near ∂B;
- a Floer trajectory from $[z^+, \gamma^+]$ to $[z^-, \gamma^-]$: $w\colon \mathbb{R} \longrightarrow S^{2M+1}$ from z^+ to z^- and $u\colon \mathbb{R} \longrightarrow C^\infty(S^1, \hat{X})$ from γ^+ to γ^- s.t.

$$\frac{\partial u}{\partial s} = -J(w, u)\Big(\frac{\partial u}{\partial t} - X_{H_w} \circ u\Big).$$

Proof of step 5.

By definition of CH and of Viterbo transfer map, and the fact that $CH^a_{n+2k-1}(X) \longrightarrow CH_{n+2k-1}(B)$ is surjective. $\square$

Updated drawing:

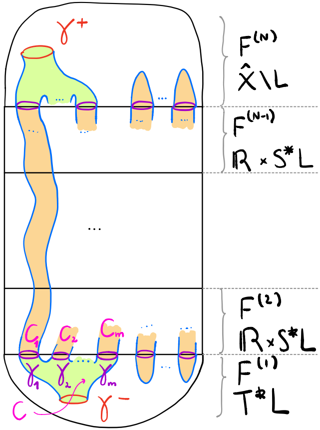


Step 6: applying SFT compactness

There exists a **broken punctured sphere** $F = (F^{(1)}, \ldots, F^{(N)})$ with $N \geq 2$ levels like the one drawn in the figure.

Proof of step 6.

The b.p.s. F is the limit of u_l as $l \to +\infty$. Recall that J_l was becoming singular on ∂W, so on the limit the u_l "break". The limit F of u_l exists by applying the **SFT compactness theorem**. We also need to do some broken holomorphic curve analysis to restrict the possibilities of limits that we obtain. □



Step 7: index computation

$m - 1 \geq k$.

Proof of step 7.

$$\begin{aligned} 0 &\leq \operatorname{ind}(C) \\ &= (n-3)(1-m) + \sum_{j=1}^{m} \mu_{CZ}(\gamma_j) - \mu_{CZ}(\gamma^-) \\ &\leq (n-3)(1-m) + \sum_{j=1}^{m} (n-1) - (n-1+2k) \\ &= 2(m-k-1). \end{aligned}$$

The first inequality is by transversality in S^1EFT, the first equality is by the index formula in SFT, and the second inequality is because $\mu_{CZ}(\gamma^-) = n+2k-1$ in step 5 by definition of CH and $\mu_{CZ}(\gamma_j) \leq n-1$ by choice of Riemannian metric on L. □

Step 8: energy computation for many disks

$\sum_{i=2}^{m} \int_{C_i} \mathrm{d}\lambda_i \leq a$.

Proof of step 8.

$$
\begin{aligned}
&\sum_{i=2}^{m} \int_{C_i} \mathrm{d}\lambda_i \\
&\quad \leq E(F) && \text{[Floer energy is additive]} \\
&\quad = \int_{\gamma^+} \lambda - \int_{\gamma^-} \lambda && [E(F) = E(u_l)] \\
&\quad \leq \int_{\gamma^+} \lambda && [\textstyle\int_{\gamma^-} \lambda = T(\gamma^-) \geq 0] \\
&\quad \leq a && [[z^+, \gamma^+] \text{ is gen. of } (FC_X^+)^a(H^+, J_l)].
\end{aligned}
$$

□

Step 9: energy computation for one disk

$\exists i = 2, \ldots, m \colon 0 < \int_{C_i} \leq \frac{a}{k}$.

Proof.

For some $i = 2, \ldots, m$

$$\begin{aligned} 0 &< \int_{C_i} \mathrm{d}\lambda_i && [C_i \text{ is holomorphic}] \\ &\leq \frac{a}{m-1} && [\text{step 8} + \text{in a set some elem. is less than avg.}] \\ &= \frac{a}{k} && [\text{by step 7: index computation}]. \end{aligned}$$

□

So, C_i is the desired disk of positive, small area. □

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Definition (cube capacity)

Define a symplectic capacity $c_P \colon \mathbf{S} \longrightarrow [0, +\infty]$ as follows. For a symplectic manifold (X, ω) and $a > 0$, consider the polydisk/cube of the same dimension as X and of radius a:

$$P(a) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \forall j = 1, \ldots, n \colon \frac{\pi |z_j|^2}{a} \leq 1 \right\}.$$

Then, we can define

$$\begin{aligned} S_P(X) &= \{ a \in \mathbb{R}^+ \mid \exists \text{ symplectic embedding } P(a) \longrightarrow X \}, \\ c_P(X, \omega) &= \sup S_P \end{aligned}$$

Lemma (comparison of cube and Lagrangian capacity)

Let (X, ω) be a symplectic manifold. Then, $c_P(X, \omega) \leq c_L(X, \omega)$.

Proof.

Define $S_L(X) = \{A_{\min}(L) \mid L \subset X \text{ is an embedded Lagrangian torus}\}$. We show that $S_P(X) \subset S_L(X)$. It suffices to assume that $a \in S_P(X)$, i.e. that there exists a symplectic embedding $\iota\colon P(a) \longrightarrow X$, and to prove that $a \in S_L(X)$, i.e. that there exists an embedded Lagrangian torus $L \subset X$ such that $a = A_{\min}(L)$. Define

$$T = \{z \in \mathbb{C}^n \mid |z_1|^2 = a/\pi, \ldots, |z_n|^2 = a/\pi\}$$

and $L = \iota(T)$. Then L is a Lagrangian torus and $A_{\min}(L, X) = A_{\min}(T, \mathbb{C}^n) = a$. We now show that $c_P(X, \omega) \le c_L(X, \omega)$.

$$\begin{aligned} c_P(X,\omega) &= \sup S_P && [\text{definition of } c_P] \\ &\le \sup S_L && [\text{proven above}] \\ &= c_L(X,\omega) && [\text{definition of } c_L]. \end{aligned}$$

$\square$

Definition (moment map)

The **moment map** is the map $\mu\colon \mathbb{C}^n \longrightarrow \mathbb{R}^n_{\geq 0}$ given by

$$\mu(z_1, \ldots, z_n) = \pi(|z_1|^2, \ldots, |z_n|^2)$$

Definition (toric domain)

A **toric domain** is a subset $X \subset \mathbb{C}^n$ of the form $X = \mu^{-1}(\Omega)$ which is a star-shaped domain.

- X is **convex** if $\hat{\Omega} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (|x_1|, \ldots, |x_n|) \in \Omega\}$ is convex.
- X is **concave** if $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex.

Proposition (consequences of the conjecture)

Let $X = \mu^{-1}(\Omega)$ *be a toric domain which is convex or concave. Let* $\delta = \max\{a \mid (a, \ldots, a) \in \Omega\}$. *Then,*

$$c_P(X) = c_L(X) = \inf_k \frac{c_k(X)}{k} = \lim_{k\to\infty} \frac{c_k(X)}{k} = \delta.$$

Proof.

$$\begin{aligned}
\delta = \lim_{k\to\infty} \frac{c_k(X)}{k} &= c_P(X) && \text{[shown in [GH18]]} \\
&\leq c_L(X) && \text{[use the previous lemma]} \\
&\leq \inf_k \frac{c_k(X)}{k} && \text{[use the conjecture]} \\
&\leq \lim_{k\to\infty} \frac{c_k(X)}{k} && [\inf \leq \lim].
\end{aligned}$$

□

References

[CM18] K. Cieliebak and K. Mohnke. Punctured holomorphic curves and lagrangian embeddings. *Invent. Math.*, 212(1):213–295, 2018.

[GH18] Jean Gutt and Michael Hutchings. Symplectic capacities from positive S^1-equivariant symplectic homology. *Algebraic & Geometric Topology*, 18(6):3537–3600, October 2018.

Thank you for listening!