Geometric quantization of the cotangent bundle of a Lie group

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Motivation - classical and quantum mechanics

Table: Comparison between classical mechanics and quantum mechanics

	classical mechanics	quantum mechanics
phase space	(M, ω)	$\mathcal{H}$
observables	$C^\infty(M, \mathbb{C})$	$\mathrm{End}(\mathcal{H})$
Lie algebra	$\{\cdot, \cdot\}$	$[\cdot, \cdot]$

So, Geometric quantization is a procedure that has input a symplectic manifold (M, ω) and has outputs

- a Hilbert space $\mathcal{H}$
- a map $Q\colon C^\infty(M, \mathbb{C}) \longrightarrow \mathrm{End}(\mathcal{H})$

We would like Q to satisfy some axioms coming from physics, the **Dirac axioms**. These are "guidelines" for $(M, \omega) \longrightarrow \mathcal{H}, Q$.

Motivation - quick outline of the construction

Geometric quantization is a construction with 3 steps:

1. Prequantization;
2. Quantization with polarizations;
3. Quantization with polarizations and half forms.

Some ideas to keep in mind:

- Each step produces an $\mathcal{H}$ and Q.
- Each step gives "better" results than the last.
- In each step, we add to M a new piece of geometric data.
- Roughly speaking,
  $\mathcal{H} <$ space of sections of some complex line bundle on M and
  for each f, $Q(f) =$ something that maps sections to sections

Line bundle

Definition

A **prequantum line bundle** for (M, ω) is a complex line bundle L, with an inner product $(\cdot, \cdot)$ and a connection ∇ which is compatible with $(\cdot, \cdot)$ (i.e. $X(s, r) = (\nabla_X s, r) + (s, \nabla_X r)$) and has curvature $-i\omega$ (i.e. $R(X, Y)s = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s = -i\omega(X, Y)s$).

Definition

(M, ω) is **quantizable** if $[\frac{\omega}{2\pi}] \in H^2(M; \mathbb{Z})$.

Theorem

There exists a prequantum line bundle for (M, ω) if and only if (M, ω) is quantizable. In this case, if M is simply connected, the prequantum line bundle is unique up to isomorphism.

Prequantization

Definition

The **prequantum Hilbert space** of (M, ω) ($\dim_{\mathbb{R}} M = 2n$) with respect to L is $\mathcal{H}(M; L) := L^2(M \longleftarrow L)$. It has the inner product

$$\langle s, r \rangle = \int_M (s, r) \frac{\omega^n}{n!}.$$

Definition

The **prequantum map** of (M, ω) with respect to L is the map Q_{pre} that for each $f \in C^\infty(M, \mathbb{C})$ associates the unbounded operator $Q_{\mathrm{pre}}(f) = i\nabla_{X_f} + f \colon \mathcal{H}(M; L) \longrightarrow \mathcal{H}(M; L)$.

Polarization

Definition

A **polarization** on M is a distribution P on $TM \otimes \mathbb{C}$ (an assignment that for each $x \in M$ gives a complex vector subspace P_x of $T_xM \otimes \mathbb{C}$) which is

1. Lagrangian, i.e., $\dim_{\mathbb{C}} P = \frac{1}{2} \dim_{\mathbb{C}} TM \otimes \mathbb{C} = n$ and for all vectors u, v in P, $\omega(u, v) = 0$;
2. Involutive, i.e., if X, Y are vector fields which lie in P, then $[X, Y]$ lies in P as well.

Polarization - types of polarizations

Definition

Let P be a polarization on M.

1. P is **real** if $P = \overline{P}$;
2. P is **complex** if $P \cap \overline{P} = \{0\}$;
3. P is **Kähler** if $P \cap \overline{P} = \{0\}$ and $\forall x \in M \colon \forall v \in P_x \colon \forall w \in \overline{P_x} \colon \; -i\omega(v, w) > 0$.

Polarization - what can we say about each type?

1. P is **real**: $P = \overline{P} \Longrightarrow P = (P \cap TM) \otimes \mathbb{C}$. $P \cap TM$ is an involutive, Lagrangian distribution. By Frobenius' theorem, there exists a **foliation** of M (partition of M into immersed submanifolds, which are the leaves of the foliation) by Lagrangian leaves L s.t. $TL = P \cap TM$.
2. P is **complex**: P Lagrangian, $P \cap \overline{P} = \{0\}$ $\Longrightarrow TM \otimes \mathbb{C} = P \oplus \overline{P}$. Define $J \colon P \oplus \overline{P} \longrightarrow P \oplus \overline{P}$ by $J(v) = iv$ if $v \in P$ and $J(v) = -iv$ if $v \in \overline{P}$. Then, J is real, i.e. $J \colon TM \longrightarrow TM$ and $J^2 = -1$. Since P is involutive, by the Newlander-Nirenberg theorem, J is integrable. So, M admits the structure of a **complex manifold** s.t. $T_{1,0}M = P$.
3. P is **Kähler**: Same as previous step. In addition, since $-i\omega(P, \overline{P}) > 0$, $\omega(\cdot, J\cdot)$ is a Riemannian metric. So, M admits the structure of a **Kähler manifold** s.t. $T_{1,0}M = P$.

Detour: complex/almost complex manifolds

Definition

Let M be a complex manifold (so it has an atlas $\mathcal{A}$ with charts $(U, x^1, \ldots, x^n, y^1, \ldots, y^n)$ s.t. transition functions satisfy the Cauchy-Riemann equations). Define $J^{\mathcal{A}}\colon TM \longrightarrow TM$ by $J^{\mathcal{A}}(\partial_x) = \partial_y$ and $J^{\mathcal{A}}(\partial_y) = -\partial_x$. (This def. is well posed)

Then, $J^{\mathcal{A}} = -1$.

Definition

An **almost complex manifold** is a pair (M, J) where M is a manifold and $J\colon TM \longrightarrow TM$ satisfies $J^2 = -1$.

Detour: complex/almost complex manifolds

$J\colon TM \otimes \mathbb{C} \longrightarrow TM \otimes \mathbb{C}$ has eigenvalues $\pm i$. Define $T_{1,0}M = (+i)$-eigenspace and $T_{0,1}M = (-i)$-eigenspace.

Definition

An almost complex structure J on M is **integrable** if there exists a complex manifold structure $\mathcal{A}$ on M such that $J = J^{\mathcal{A}}$.

Theorem (Newlander-Nirenberg)

Let (M, J) be an almost complex manifold. J is integrable if and only if $T_{1,0}M$ is involutive.

Quantization without half forms - Hilbert space

Definition

A section s of L is P-**polarized** if $\forall X \in \mathfrak{X}(\overline{P})$: $\nabla_X s = 0$.

Definition

Define the **quantum Hilbert space**, denoted $\mathcal{H}(M; L, P)$, as the closure inside $L^2(M \longleftarrow L)$ of the set of smooth, square integrable, polarized sections.

Quantization without half forms - quantum map

Definition

Let $f \in C^\infty(M, \mathbb{C})$. f is **quantizable** if $Q_{\mathrm{pre}}(f)$ maps the space of smooth polarized sections to itself.

Definition

The **quantum map** of (M, ω) with respect to L, P is the map Q_{pre} that for each $f \in C^\infty(M, \mathbb{C})$ which is quantizable associates the unbounded operator $Q_{\mathrm{pre}}(f) = i\nabla_{X_f} + f \colon \mathcal{H}(M; L, P) \to \mathcal{H}(M; L, P)$.

$$
\begin{array}{lccc}
\text{previous quantum map} & \mathcal{H}(M;L) & \xrightarrow{Q_{pre}(f)} & \mathcal{H}(M;L) \\
 & \uparrow & & \uparrow \\
\text{new quantum map} & \mathcal{H}(M;L,P) & \xrightarrow[Q_{pre}(f)]{} & \mathcal{H}(M;L,P)
\end{array}
$$

Quantization with half forms - new line bundles

Definition

The **canonical bundle** of P is the complex line bundle $\mathcal{K}_P$ over M ($\dim_{\mathbb{R}} M = 2n$) whose fibre above $x \in M$ is

$$\mathcal{K}_P|_x = \left\{ \alpha \in \bigwedge_{k=1}^{n} T_x^* M \otimes \mathbb{C} \;\middle|\; \forall v \in \overline{P_x} \colon \iota_v \alpha = 0 \right\}.$$

Definition

A **square root** of $\mathcal{K}_P$ is a pair (δ_P, ϕ_P), where δ_P is a complex line bundle and $\phi_P \colon \delta_P \otimes \delta_P \longrightarrow \mathcal{K}_P$ is a complex vector bundle iso..

For X in a certain subset of $\mathfrak{X}(M)$, have
$L_X \colon C^\infty(M \longleftarrow \mathcal{K}_P) \longrightarrow C^\infty(M \longleftarrow \mathcal{K}_P)$ and
$L_X \colon C^\infty(M \longleftarrow \delta_P) \longrightarrow C^\infty(M \longleftarrow \delta_P)$.

Quantization with half forms - $\mathcal{H}$ and Q

If P is Kähler, then the line bundle $L \otimes \delta_{\mathbb{C}}$ admits a canonical Hermitian inner product $(\cdot, \cdot)$ and a canonical (partial) connection ∇.

Definition

The **half form Hilbert space**, denoted $\mathcal{H}(M; L, P, \delta_P)$, is the closure inside $L^2(M \longleftarrow L \otimes \delta_P)$ of the set of smooth, square integrable, polarized sections.

Definition

The **half form quantum map** of (M, ω) with respect to L, P, δ_P is the map Q that for each $f \in C^\infty(M, \mathbb{C})$ which is quantizable associates the unbounded operator

$$Q(f)(\mu \otimes \nu) = (Q_{\mathrm{pre}}(f)\mu) \otimes \nu + \mu \otimes iL_{X_f}\nu.$$

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Main ideas

Joint work with José Mourão and João Nunes, in [MNP19].

Main ideas - setup that we consider

- Let G be a Lie group. Then, T^*G is a symplectic manifold. We are going to apply the previous procedure to T^*G.
- In this case, we can choose $L = T^*G \times \mathbb{C}$. It remains to choose the polarization.
- We define a family of polarizations, one $P_{\tau,\sigma}$ for each $\tau, \sigma \in \mathbb{C}$.

Main ideas - goals

1. Study the polarizations and see what type of geometric structures exist.
2. Study the resulting Hilbert spaces.

Setup - Lie group

Assumption - Lie group

- Let G be a Lie group which is compact and connected.
- Let $\mathfrak{g}$ denote the Lie algebra of G.
- Let $\langle\cdot,\cdot\rangle$ be an Ad-invariant inner product on $\mathfrak{g}$.

Assumption - torus/algebra

- Let T be a maximal torus in G. So T is an abelian, compact, connected Lie subgroup which is maximal.
- Let $\mathfrak{t}$ denote the Lie algebra of T, which is a Lie subalgebra of $\mathfrak{g}$.
- So, $\mathfrak{t}$ is a maximal abelian subalgebra.

Setup - prequantum line bundle

As we saw, (T^*G, ω) is a symplectic manifold with exact symplectic form, $\omega = \mathrm{d}\theta$. The following set of data is a prequantum line bundle for T^*G:

Assumption - prequantum line bundle

Let $(L, (\cdot, \cdot), \nabla)$ be the following prequantum line bundle:

- $L = T^*G \times \mathbb{C}$
- $(s, r) = s\bar{r}$
- $\nabla_X s = X(s) - i\theta(X)s$

Setup - Lie group actions

Action of $G \times T$ on G

$G \times T$ acts on G, by

$$\begin{aligned}(G \times T) \times G &\longrightarrow G\\ ((g,f),h) &\longmapsto (g,f)h := A_{(g,f)}h := ghf^{-1}.\end{aligned}$$

Action of $G \times T$ on T^*G

The previous action induces an action of $G \times T$ on T^*G, given by

$$\begin{aligned}(G \times T) \times T^*G &\longrightarrow T^*G\\ ((g,t),\alpha) &\longmapsto T^*A_{(g^{-1},t^{-1})}\alpha.\end{aligned}$$

Setup - Hamiltonian functions

Assumption - Hamiltonian functions

Let $f, h\colon T^*G \longrightarrow \mathbb{R}$ be functions satisfying:

- h is $G \times G$-invariant;
- f is $G \times T$-invariant;
- Some other assumptions on f and h.

Hamiltonian vector fields

The functions f, h have Hamiltonian vector fields, uniquely determined by $\mathrm{d}f = \omega(X_f, \cdot)$, $\mathrm{d}h = \omega(X_h, \cdot)$.

Hamiltonian flows

Denote by $\phi^t_{X_h}, \phi^s_{X_f} \colon T^*G \longrightarrow T^*G$ the Hamiltonian flows of h, f.

Polarizations - $t, s \in \mathbb{R}$

Vertical polarization

- For each $(g, \alpha) \in T^*G$ (so, $\alpha \in T_g^*G$), consider the map $\mathrm{D}\pi(g, \alpha)\colon T_{(g,\alpha)}(T^*G) \longrightarrow T_g G$.
- Define $P_{0,0}|_{(g,\alpha)} = \ker \mathrm{D}\pi(g, \alpha) \otimes \mathbb{C} < T_{(g,\alpha)}(T^*G) \otimes \mathbb{C}$.
- P is a polarization, called the **vertical polarization**.

Definition of the family of polarizations for $t, s \in \mathbb{R}$

$$P_{t,s}|_{(g,\alpha)} = \mathrm{D}(\phi^t_{X_h} \circ \phi^s_{X_f})|_{(\phi^t_{X_h} \circ \phi^s_{X_f})^{-1}(g,\alpha)} P_{0,0}|_{(\phi^t_{X_h} \circ \phi^s_{X_f})^{-1}(g,\alpha)}$$

Polarizations - $\tau, \sigma \in \mathbb{C}$

By left translations, $T^*G \cong G \times \mathfrak{g}^*$. Using the inner product of $\mathfrak{g}$, $\mathfrak{g}^* \cong \mathfrak{g}$. So, $T_{(g,\alpha)}(T^*G) \cong \mathfrak{g} \oplus \mathfrak{g}$.

Computation of $P_{t,s}$

As a subspace of $\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$, for every $x, y \in G \times \mathfrak{g}$,

$$P_{t,s}|_{(x,y)} = \left\{ (T_{t,s}A, A) \;\middle|\; A \in \mathfrak{g}_{\mathbb{C}} \right\}.$$

where $T_{t,s} \colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is a linear map (with an explicit formula).

Definition of the family of polarizations for $\tau, \sigma \in \mathbb{C}$

Define $P_{\tau,\sigma}|_{(x,y)}$ by replacing $t \longrightarrow \tau$, $s \longrightarrow \sigma$:

$$P_{\tau,\sigma}|_{(x,y)} = \left\{ (T_{\tau,\sigma}A, A) \;\middle|\; A \in \mathfrak{g}_{\mathbb{C}} \right\},$$

Results - Kähler structures

Theorem ([MNP19])

1. *$P_{\tau,\sigma}$ is invariant under the action of $G \times T$ on T^*G.*
2. *For $\mathrm{Im}\tau, \mathrm{Im}\sigma > 0$, $P_{\tau,\sigma}$ is a Kähler polarization. In particular, T^*G has the structure of a Kähler manifold for which $T_{1,0}(T^*G) = P_{\tau,\sigma}$.*
3. *For $\mathrm{Im}\tau, \mathrm{Im}\sigma > 0$, T^*G has a global Kähler potential (a function κ s.t. $i\partial\bar{\partial}\kappa = \omega$).*
4. *The action of $G \times T$ on T^*G is by Kähler isomorphisms.*

Results - Hilbert spaces

There exists a canonical linear map $U_{\tau,\sigma} \colon \mathcal{H}_{0,0} \longrightarrow \mathcal{H}_{\tau,\sigma}$ (not just in our case - in general, not ness. a Unitary iso.). There is a natural action of $G \times T$ on $\mathcal{H}(T^*G; L, P_{\tau,\sigma}, \delta_{P_{\tau,\sigma}})$.

1. If $\mathrm{Im}\tau > 0, \mathrm{Im}\sigma > 0$, we give an explicit computation of $\mathcal{H}_{\tau,\sigma} = \mathcal{H}(T^*G; L, P_{\tau,\sigma}, \delta_{P_{\tau,\sigma}}) =$ (a big expression); (what we have to compute is what are the polarized sections)
2. If $\mathrm{Im}\tau > 0, \mathrm{Im}\sigma > 0$, $U_{\tau,\sigma}$ is $G \times T$-equivariant and a linear isomorphism;
3. If $\tau = 0, \mathrm{Im}\sigma > 0$, then $U_{\tau,\sigma}$ is a unitary isomorphism.

References

[MNP19] José M. Mourão, João P. Nunes, and Miguel B. Pereira. Partial coherent state transforms, $G \times T$-invariant Kähler structures and geometric quantization of cotangent bundles of compact Lie groups. *arXiv:1907.05232 [math-ph]*, September 2019.

Thank you for listening!