S^1-equivariant symplectic homology and symplectic capacities

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Theorem (comparison between c_L and c_k, work in progress)

If (X, λ) is a Liouville domain, $\pi_1(X) = \{0\}$ and $c_1(TX)|_{\pi_2(X)} = 0$, then

$$c_L(X, \lambda) \leq \inf_{k \in \mathbb{N}} \frac{c_k(X, \lambda)}{k}.$$

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Definition (symplectic manifold)

A **symplectic manifold** is a pair (M, ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate. An **exact symplectic manifold** is a pair (M, θ) such that $(M, \mathrm{d}\theta)$ is a symplectic manifold.

Definition (Liouville domain)

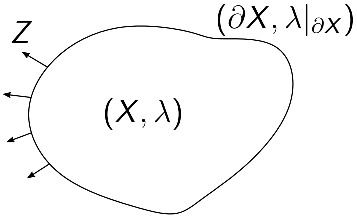
A **Liouville domain** is a pair (X, λ), where X is a compact, connected smooth manifold with boundary ∂X and $\lambda \in \Omega^1(X)$ is such that $\mathrm{d}\lambda \in \Omega^2(X)$ is symplectic, $\lambda|_{\partial X}$ is contact and the orientations on ∂X coming from $(X, \mathrm{d}\lambda)$ and coming from $\lambda|_{\partial X}$ are equal.

Definition (Liouville vector field)

If (X, λ) is a Liouville domain, it's **Liouville vector field** is the unique vector field Z such that $\lambda = \iota_Z \mathrm{d}\lambda$.

Lemma (Z is outward pointing)

If (X, λ) is a Liouville domain, then Z is outward pointing at ∂X.



Definition (types of morphisms for symplectic manifolds)

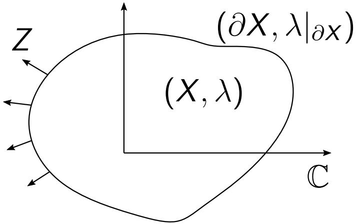
Let (X, ω_X), (Y, ω_Y) be symplectic manifolds (possibly with boundary and corners) and $\varphi\colon X \longrightarrow Y$ be an embedding. Then, φ is **symplectic** if $\varphi^*\omega_Y = \omega_X$. A **symplectomorphism** is a symplectic embedding which is a diffeomorphism. If (X, λ_X), (Y, λ_Y) are exact symplectic, then we say that

1. φ is **symplectic** if $\varphi^*\lambda_Y - \lambda_X$ is closed ($\Longleftrightarrow$ to previous def.);
2. φ is **exact symplectic** if $\varphi^*\lambda_Y - \lambda_X$ is exact;
3. φ is **Liouville** if $\varphi^*\lambda_Y - \lambda_X = 0$.

Definition (star-shaped domain)

A **star-shaped domain** is a compact, connected $2n$-dimensional submanifold X of $\mathbb{C}^n$ with boundary ∂X such that (X, λ) is a Liouville domain, where

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} \Big(y^i \mathrm{d} x^i - x^i \mathrm{d} y^i \Big).$$



Definition (moment map)

The **moment map** is the map $\mu\colon \mathbb{C}^n \longrightarrow \mathbb{R}^n_{\geq 0}$ given by

$$\mu(z_1, \ldots, z_n) = \pi(|z_1|^2, \ldots, |z_n|^2)$$

Definition (toric domain)

A **toric domain** is a star-shaped domain X of the form $X = \mu^{-1}(\Omega)$.

- X is **convex** if $\hat{\Omega} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (|x_1|, \ldots, |x_n|) \in \Omega\}$ is convex.
- X is **concave** if $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex.

Definition (diagonal of a toric domain)

If $X = \mu^{-1}(\Omega)$ is toric, define $\delta(X) := \max\{a \mid (a, \ldots, a) \in \Omega\}$.

Example (toric domains)

- **Ellipsoid:**

$$E(a_1, \dots, a_n) = \mu^{-1}(\Omega_E(a_1, \dots, a_n))$$
$$\Omega_E(a_1, \dots, a_n) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{j=1}^{n} \frac{x_j}{a_j} \leq 1 \right\}$$

- **Ball:** $B(a) = E(a, \dots, a)$
- **Cylinder:** $Z(a) = E(a, \infty, \dots, \infty)$
- **Cube:**

$$P(a) = \mu^{-1}(\Omega_P(a))$$
$$\Omega_P(a) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \forall j = 1, \dots, n\colon \frac{x_j}{a} \leq 1 \right\}$$

Definition (symplectic capacity)

A **domain for a symplectic capacity** is a subcategory $\mathbf{D}$ of the category of symplectic manifolds such that $(M,\omega) \in \mathbf{D}$ implies $(M,\alpha\omega) \in \mathbf{D}$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. A **symplectic capacity** is a map $c \colon \mathbf{D} \longrightarrow [0,+\infty]$, such that

(Monotonicity) c is a functor, i.e. if $(M,\omega_M) \longrightarrow (N,\omega_N)$ is a morphism in $\mathbf{D}$ then $c(M,\omega_M) \leq c(N,\omega_N)$.

(Conformality) For every $\alpha \in \mathbb{R} \setminus \{0\}$ and $(M,\omega) \in \mathbf{D}$ we have that $c(M,\alpha\omega) = |\alpha| c(M,\omega)$.

If $B(1), Z(1) \in \mathbf{D}$, then c is **nontrivial** or **normalized** if (resp.):

(Nontriviality) $0 < c(B(1)) \leq c(Z(1)) < +\infty$.

(Normalization) $0 < c(B(1)) = 1 = c(Z(1)) < +\infty$.

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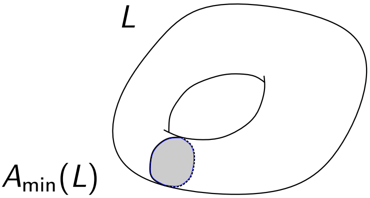
Definition (minimal symplectic area of a Lagrangian submanifold)

Let (X, ω) be a symplectic manifold. If L is a Lagrangian submanifold of X, then we define the **minimal symplectic area of** L, $A_{\mathrm{min}}(L)$, by

$$\begin{aligned} A_{\mathrm{min}}(L) &:= \inf \{\omega(\sigma) \mid \sigma \in \pi_2(X, L),\ \omega(\sigma) > 0\} \\ &= \inf \left\{ \int_D u^*\omega \,\middle|\, u \colon (D, \partial D) \longrightarrow (X, L),\ \int_D u^*\omega > 0 \right\} \\ &\in [0, \infty]. \end{aligned}$$



Definition (Lagrangian capacity)

We define the **Lagrangian capacity of** (X, ω), $c_L(X, \omega)$, by

$$c_L(X, \omega) \coloneqq \sup\{A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus}\} \in [0, \infty].$$

Proposition (Properties of the Lagrangian capacity)

The Lagrangian capacity c_L satisfies:

(Monotonicity) If $\iota \colon (X, \omega) \longrightarrow (X', \omega')$ is a symplectic embedding s.t. $\pi_2(X', \iota(X)) = 0$, then $c_L(X, \omega) \leq c_L(X', \omega')$.

(Conformality) For all $\alpha \in \mathbb{R} \backslash \{0\}$, $c_L(X, \alpha\omega) = |\alpha| c_L(X, \omega)$.

Definition (cube capacity)

We define the **cube capacity of** (X, ω), $c_P(X, \omega)$, by

$$c_P(X, \omega) = \sup\{a \in \mathbb{R}_{>0} \mid \exists \text{ symplectic embedding } P(a) \longrightarrow X\}.$$

Lemma (comparison of cube and Lagrangian capacity)

Let (X, ω) be a symplectic manifold. Then, $c_L(X, \omega) \geq c_P(X, \omega)$.

Proof.

Since

$$\begin{aligned} c_L(X, \omega) &= \sup\{A_{\min}(L) \mid L \subset X \text{ embedded Lagrangian torus}\}, \\ c_P(X, \omega) &= \sup\{a \in \mathbb{R}_{>0} \mid \exists \text{ symplectic embedding } P(a) \longrightarrow X\}, \end{aligned}$$

it suffices to assume that $a \in \mathbb{R}_{>0}$ is such that there exists a symplectic embedding $P(a) \longrightarrow X$ and to prove that there exists an embedded Lagrangian torus $L \subset X$ such that $a = A_{\min}(L)$. Define

$$T = \{z \in \mathbb{C}^n \mid |z_1|^2 = a/\pi, \ldots, |z_n|^2 = a/\pi\}$$

and $L = \iota(T)$. Then L is as desired. □

Lemma (comparison of cube capacity and δ)

If X is a convex or concave toric domain, then $c_P(X) \geq \delta(X)$.

Proof.

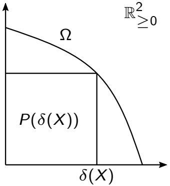
X is a convex or concave toric domain

$$\implies P(\delta(X)) \subset X$$

$$\implies \delta(X) \in \{a \in \mathbb{R}_{>0} \mid \exists \text{ symplectic embedding } P(a) \to X\}$$

$$\implies \delta(X) \leq c_P(X).$$

□



Proposition (Lagrangian capacity of the ball, [CM18])

$c_L(B(1)) = 1/n$.

Proof.

$(\geq)$: $c_L(B(1)) \geq c_P(B(1)) \geq \delta(B(1)) = 1/n$.
$(\leq)$: This is hard. Uses the main theorem of [CM18], which says that there are disks with boundary on a Lagrangian of small area, and it's proof uses holomorphic curve techniques. □

Proposition (Lagrangian capacity of the cylinder, [CM18])

$c_L(Z(1)) = 1.$

Proof.

$(\geq)$: $c_L(Z(1)) \geq c_P(Z(1)) \geq \delta(Z(1)) = 1.$
$(\leq)$: This is hard. Uses the concepts of Hofer norm, Hofer energy, displacement energy, and a result of Chekanov comparing $A_{\min}$ and displacement energy. See [CM18, HZ11, Che98]. □

Remark (motivation for conjecture)

Let X be a convex or concave toric domain. We have proven that $c_L(X) \geq \delta(X)$. For the ball and the cylinder, [CM18] have proven that $c_L(X) = \delta(X)$. This motivates the conjecture below.

Conjecture (Lagrangian capacity of ellipsoid, [CM18])

Let $E(a_1, \ldots, a_n) \subset \mathbb{C}^n$ be an ellipsoid. Then,

$$c_L(E(a_1, \ldots, a_n)) = \delta(E(a_1, \ldots, a_n)) = \left(\frac{1}{a_1} + \cdots \frac{1}{a_n} \right)^{-1}.$$

Remark (main theorem $\Longrightarrow$ conjecture)

Using our main theorem, we will actually show that $c_L(X) = \delta(X)$ for any convex or concave toric domain (in the section about consequences of the main theorem).

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Let (X, λ) be a nondegenerate Liouville domain.

Definition (completion)

The **completion** of (X, λ) is an exact symplectic manifold $(\hat{X}, \hat{\lambda})$ given as follows. As a manifold, $\hat{X} = X \cup_{\partial X} \mathbb{R}_{\geq 0} \times \partial X$ (where we glue with the flow of the Liouville vector field). The form $\hat{\lambda}$ is given by

$$\hat{\lambda}|_X = \lambda$$
$$\hat{\lambda}|_{\mathbb{R}_{\geq 0} \times \partial X} = e^r \lambda|_{\partial X}.$$

Consider a “suitable” function $\hat{f}_q \colon S^{2q+1} \longrightarrow \mathbb{R}$.

Definition (Positive S^1-equivariant Floer complex)

- The PS^1EFC of (X, λ) w.r.t. $H \colon S^1 \times S^{2q+1} \times \hat{X} \longrightarrow \mathbb{R}$ and an almost complex structure $J \colon S^1 \times S^{2q+1} \times \hat{X} \longrightarrow \mathrm{End}(T\hat{X})$ is a chain complex of $\mathbb{Q}$-modules denoted by $FC^+(X, \lambda, H, J)$.
- $FC^+(X, \lambda, H, J)$ is generated by (S^1-eq. classes of) tuples $\boldsymbol{\gamma} = (z, \gamma)$, where $z \in S^{2q+1}$ is a critical point of $\hat{f}_q$ and γ is a 1-periodic orbit of H_z.
- The differential of $FC^+(X, \lambda, H, J)$ counts (S^1-eq. classes of) Floer trajectories $\mathbf{u} = (w, u)$, where $w \colon \mathbb{R} \longrightarrow S^{2q+1}$ is a gradient flow line of $\hat{f}_q$ and $u \colon \mathbb{R} \times S^1 \longrightarrow \hat{X}$ is a sol. of the Floer eq.

$$\frac{\partial u}{\partial s} = -J(t, w, u)\left(\frac{\partial u}{\partial t} - X_H(t, w, u)\right).$$

Definition (Positive S^1-equivariant Floer homology)

$FH^+(X, \lambda, H, J) = H(FC^+(X, \lambda, H, J))$.

Definition (Positive S^1-equivariant symplectic homology)

$CH(X, \lambda) = \varinjlim_{H,J} FH^+(X, \lambda, H, J)$.

Remark (Properties of $CH(X, \lambda)$)

- Action filtration: $\iota^a \colon CH^a(X, \lambda) \longrightarrow CH(X, \lambda)$;
- Viterbo transfer maps: if $\phi \colon (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ is an exact symplectic embedding, there exists a corresponding **Viterbo transfer map** $\phi_! \colon CH(Y, \lambda_Y) \longrightarrow CH(X, \lambda_X)$.

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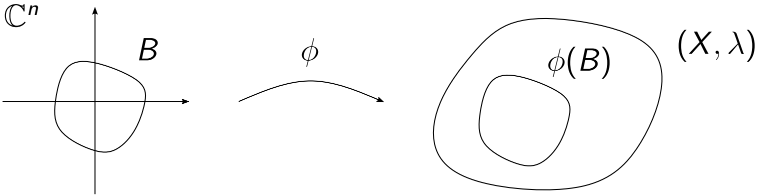
Definition (Gutt-Hutchings capacity)

Let (X, λ) be a Liouville domain (nondegenerate, satisfying the same topological assumptions) and $k \in \mathbb{N}$. The ***k*th Gutt-Hutchings capacity** of (X, λ), denoted $c_k(X, \lambda)$, is given as follows. Choose $B \subset \mathbb{C}^n$ a nondegenerate star-shaped domain and $\phi \colon B \longrightarrow X$ an exact symplectic embedding. Then, $c_k(X, \lambda)$ is the infimum over $a > 0$ such that the following map is surjective:

$$CH^a_{n+2k-1}(X, \lambda) \xrightarrow{\iota^a} CH_{n+2k-1}(X, \lambda) \xrightarrow{\phi_!} CH_{n+2k-1}(B, \lambda_0)$$



Remark (standard vs alternative definition of c_k)

Actually, the definition we gave of c_k is an alternative definition. The standard definition was given in [GH18].

- **Standard definition** ([GH18]): relies on additional properties of positive S^1-equivariant symplectic homology (which we did not mention), namely maps U and δ. This definition doesn't rely on choosing B.
- **Alternative definition**: it's possible to prove that the definition we gave and the one given in [GH18] are equivalent. Our definition doesn't depend on the maps U or δ, but depends on choosing B. We will only need the alternative definition to understand the proof of the main theorem.

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Proposition (consequences of the main theorem)

If $X = \mu^{-1}(\Omega)$ is a convex or concave toric domain,

$$c_P(X) = c_L(X) = \inf_k \frac{c_k(X)}{k} = \lim_{k\to\infty} \frac{c_k(X)}{k} = \delta(X).$$

Proof.

$$\begin{aligned} \delta(X) = \lim_{k\to\infty} \frac{c_k(X)}{k} &= c_P(X) && \text{[shown in [GH18]]} \\ &\leq c_L(X) && \text{[use a previous lemma]} \\ &\leq \inf_k \frac{c_k(X)}{k} && \text{[use the main theorem]} \\ &\leq \lim_{k\to\infty} \frac{c_k(X)}{k} && [\inf \leq \lim]. \end{aligned}$$

□

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We now present what hopefully will be the proof of the main theorem (we present a version of the proof with some imprecisions to make the discussion simpler, but the key ideas are here).

Step 1/8: what we need to assume and prove

It suffices to assume that

- (X, λ) is a ndg. Liouville domain, $\pi_1(X) = 0$, $c_1(TX)|_{\pi_2(X)} = 0$
- $k \in \mathbb{N}$
- $L \subset \operatorname{int} X$ is an embedded Lagrangian torus
- $a > c_k(X, \lambda)$

and to prove that there exists $\sigma \in \pi_2(X, L)$ s.t. $0 < \omega(\sigma) \leq a/k$.

Proof of step 1.

By definition of c_L. □

Step 2/8: constructing a tubular neighbourhood

There exists g a Riemannian metric on L, $W \subset \operatorname{int} X$ a closed set containing L and a symplectomorphism $\psi \colon W \longrightarrow D^* L$ and such that for every closed geodesic γ of L, if $l(\gamma) \leq a$ then γ is noncontractible and nondegenerate and $0 \leq \operatorname{ind}_{\mathrm{M}}(\gamma) \leq n - 1$.

Proof of step 2.

By the Lagrangian neighbourhood theorem plus a lemma from [CM18] which says that metrics of nonpositive sectional curvature (for example the flat metric on the torus) can be perturbed to have the desired property. □

Step 3/8: choosing a small ball inside W

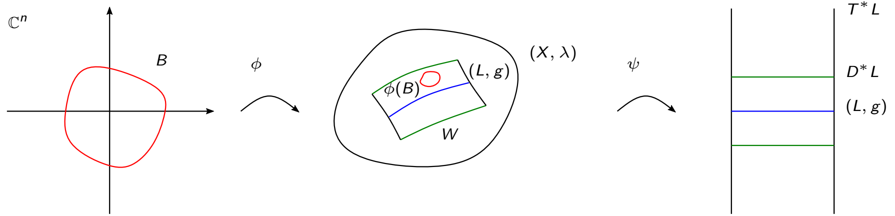
There exists $B \subset \mathbb{C}^n$ a nondegenerate star-shaped domain and $\phi\colon B \longrightarrow X$ an exact symplectic embedding such that $\phi(B) \subset \operatorname{int} W$ and the following map is surjective:

$$CH^a_{n+2k-1}(X, \lambda) \xrightarrow{\iota^a} CH_{n+2k-1}(X) \xrightarrow{\phi_!} CH_{n+2k-1}(B, \lambda_0)$$

Proof of step 3.

By definition of $c_k(X)$ and because $c_k(X) < a$. □

So, until now we have the following:



Remark (Next steps)

Recall that we wish to show that there exists a disk with boundary on L and of small area. To accomplish that, we will

- Create a special sequence of Floer trajectories;
- Take the limit of those Floer trajectories, which is a "Floer trajectory" which is "broken" into many components;
- One of those components is the disk we want.

Step 4/8: choosing auxiliary data

Choose H a Hamiltonian and J an almost complex structure on $\hat{X}$. Do a construction from SFT (**symplectic field theory**) called **neck stretching**, which produces a sequence of a.c.s. $(J_m)_{m \in \mathbb{N}}$ on $\hat{X}$.

Step 5/8: applying the definition of *CH*

For every $m \in \mathbb{N}$ there exist generators $\gamma_m^{\pm}$ of $FC^+(X, H, J_m)$ and a Floer trajectory $\mathbf{u}_m$ from γ_m^+ to γ_m^- such that γ_m^+ is near ∂X, γ_m^- is near ∂B, $\mathcal{A}_H(\gamma_m^+) \leq a$ and $\mu(\gamma_m^-) \geq n + 2k - 1$.

Proof of step 5.

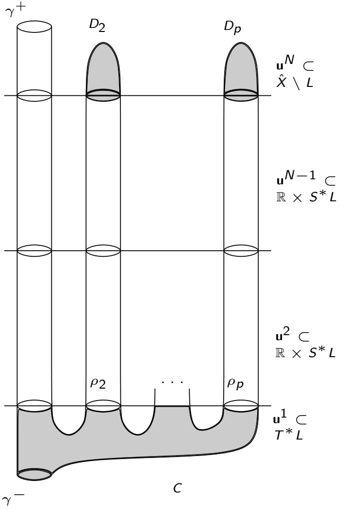
By definition of CH and of Viterbo transfer map, and the fact that $CH^a_{n+2k-1}(X, \lambda) \longrightarrow CH_{n+2k-1}(B, \lambda_0)$ is surjective. □

Step 6/8: applying SFT compactness

There exists a **broken punctured sphere** $\mathbf{u} = (\mathbf{u}^1, \ldots, \mathbf{u}^N)$ with $N \geq 2$ levels like the one drawn in the figure.

Proof of step 6.

There is a uniform energy bound $E(\mathbf{u}_m) \leq a$. The b.p.s. $\mathbf{u}$ is the limit of $\mathbf{u}_m$ as $m \to +\infty$. The limit exists by applying a modification of the **SFT compactness theorem**. We also need to do some broken holomorphic curve analysis to restrict the possibilities of limits that we obtain. □



Step 7/8: index computation

$p - 1 \geq k$.

Proof of step 7.

$$
\begin{aligned}
0 &\leq \operatorname{ind}(C) \\
&= (n-3)(1-p) + \sum_{j=1}^{p} \mu_{CZ}(\rho_j) - \mu(\gamma^-) \\
&\leq (n-3)(1-p) + \sum_{j=1}^{p} (n-1) - (n-1+2k) \\
&= 2(p-k-1).
\end{aligned}
$$

The first inequality is by transversality in S^1EFT, the first equality is by the index formula in S^1EFT, and the second inequality is because $\mu(\gamma^-) = n + 2k - 1$ in step 5 by definition of CH and $\mu_{CZ}(\rho_j) \leq n - 1$ by choice of Riemannian metric on L. □

Step 8/8: energy computation

$\exists i \in \{2, \ldots, p\} \colon 0 < E(D_i) \leq a/k$.

Proof.

By definition of average, there exists $i \in \{2, \ldots, p\}$ such that

$$\begin{aligned} E(D_i) &\leq \frac{E(D_2) + \cdots + E(D_p)}{p-1} && \text{[by definition of average]} \\ &= \frac{E(D_2 \cup \cdots \cup D_p)}{p-1} && \text{[energy is additive]} \\ &\leq \frac{a}{p-1} && \text{[by the uniform energy bound]} \\ &\leq \frac{a}{k} && \text{[by the index computation].} \end{aligned}$$

□

So, D_i is the desired disk of positive, small area. □

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Thank you for listening!