The Lagrangian capacity of toric domains

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Conjecture ([Per22, Conjecture 6.24])

If X_Ω is a convex or concave toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Goal

To **motivate** and **prove** the conjecture (in some special cases).

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Definition 2.1

The **moment map** is the map $\mu\colon \mathbb{C}^n \longrightarrow \mathbb{R}^n_{\geq 0}$ given by

$$\mu(z_1, \dots, z_n) \coloneqq \pi(|z_1|^2, \dots, |z_n|^2).$$

Define also

$$\begin{aligned}
\Omega_X &\coloneqq \Omega(X) \coloneqq \mu(X) \subset \mathbb{R}^n_{\geq 0}, && \text{for every } X \subset \mathbb{C}^n,\\
X_\Omega &\coloneqq X(\Omega) \coloneqq \mu^{-1}(\Omega) \subset \mathbb{C}^n, && \text{for every } \Omega \subset \mathbb{R}^n_{\geq 0},\\
\delta_\Omega &\coloneqq \delta(\Omega) \coloneqq \sup\{a \mid (a, \dots, a) \in \Omega\}, && \text{for every } \Omega \subset \mathbb{R}^n_{\geq 0}.
\end{aligned}$$

We call δ_Ω the **diagonal** of Ω.

Definition 2.2

A **toric domain** is a star-shaped domain X of the form $X = X_\Omega$. A toric domain $X = X_\Omega$ is

- **convex** if $\hat{\Omega} \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (|x_1|, \dots, |x_n|) \in \Omega\}$ is convex;
- **concave** if $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex.

Example 2.3

The following are toric domains:

$$E(a_1, \ldots, a_n) \coloneqq \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{\pi |z_j|^2}{a_j} \leq 1 \right\} \qquad \textbf{(ellipsoid)}$$

$$B(a) \coloneqq \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{\pi |z_j|^2}{a} \leq 1 \right\} \qquad \textbf{(ball)}$$

$$Z(a) \coloneqq \{ z \in \mathbb{C}^n \mid \pi |z_1|^2 \leq a \} \qquad \textbf{(cylinder)}$$

$$P(a) \coloneqq \{ z \in \mathbb{C}^n \mid \forall j = 1, \ldots, n \colon \pi |z_j|^2 \leq a \} \qquad \textbf{(cube)}$$

$$N(a) \coloneqq \{ z \in \mathbb{C}^n \mid \exists j = 1, \ldots, n \colon \pi |z_j|^2 \leq a \}$$

(nondisjoint union of cylinders)

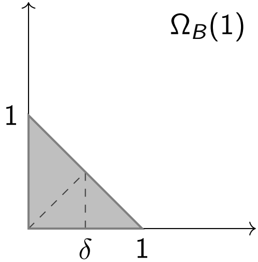

Figure: Ball and cube

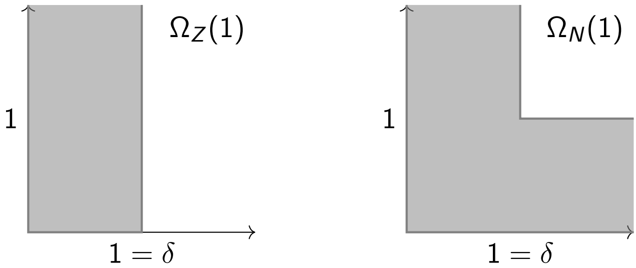

Figure: Cylinder and nondisjoint union of cylinders

Definition 2.4

A **symplectic capacity** is a map c which to every symplectic manifold (possibly in a restricted subclass) assigns an element of $[0, +\infty]$, such that

- (Monotonicity) If $(X, \omega_X) \longrightarrow (Y, \omega_Y)$ is a symplectic embedding of codimension 0 (possibly in a restricted subclass) then $c(X, \omega_X) \leq c(Y, \omega_Y)$;
- (Conformality) If $\alpha > 0$ then $c(X, \alpha\omega) = \alpha c(X, \omega)$.

Example 2.5

The **cube capacity** is given by

$$c_P(X, \omega) \coloneqq \sup\{a \mid \exists \text{ symplectic embedding } P^{2n}(a) \longrightarrow X\}.$$

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Definition 3.1 ([CM18, Section 1.2])

Let (X, ω) be a symplectic manifold. If L is a Lagrangian submanifold of X, then we define the **minimal symplectic area of** L by

$$A_{\min}(L) := \inf\{\omega(\sigma) \mid \sigma \in \pi_2(X, L),\ \omega(\sigma) > 0\}.$$

Definition 3.2 ([CM18, Section 1.2])

The **Lagrangian capacity** of (X, ω) is

$$c_L(X) := \sup\{A_{\min}(L) \mid L \subset X \text{ is an embedded Lagrangian torus}\}.$$

Lemma 3.3

If X is a star-shaped domain, then $c_L(X) \geq c_P(X)$.

Proof.

Let $\iota\colon P(a) \longrightarrow X$ be a symplectic embedding, for some $a > 0$. We want to show that $c_L(X) \geq a$. Define $T := \mu^{-1}(a, \ldots, a) \subset \partial P(a)$ and $L := \iota(T) \subset X$. Then, $c_L(X) \geq A_{\mathrm{min}}(L) = A_{\mathrm{min}}(T) = a$. $\square$

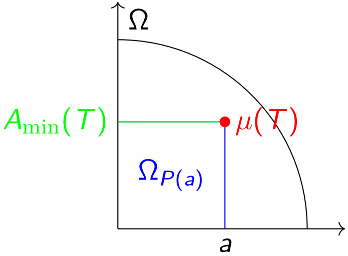

Figure: Proof of $c_L(X) \geq c_P(X)$ for $X = X_\Omega$

Lemma 3.4

If X_Ω is a convex or concave toric domain, then $c_P(X_\Omega) \geq \delta_\Omega$.

Proof.

Since X_Ω is convex or concave, we have $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$. The result follows since $c_P(X_\Omega) := \sup\{a \mid \exists\, P(a) \hookrightarrow X_\Omega\}$. □

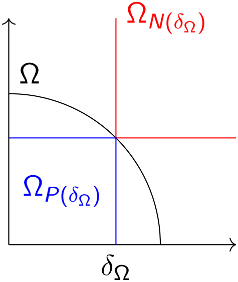

Figure: If X_Ω is convex or concave then $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$

We now consider the results by Cieliebak–Mohnke for the Lagrangian capacity of the ball and the cylinder.

Proposition 3.5 ([CM18, Corollary 1.3])

The Lagrangian capacity of the ball is

$$c_L(B^{2n}(1)) = \frac{1}{n} = \delta_{\Omega(B^{2n}(1))}.$$

Proposition 3.6 ([CM18, p. 215-216])

The Lagrangian capacity of the cylinder is

$$c_L(Z^{2n}(1)) = 1 = \delta_{\Omega(Z^{2n}(1))}.$$

Conclusion

$$\begin{aligned} X_\Omega \text{ is a convex or concave toric domain} &\Longrightarrow c_L(X_\Omega) \geq \delta_\Omega \\ X_\Omega \text{ is the ball or the cylinder} &\Longrightarrow c_L(X_\Omega) = \delta_\Omega \end{aligned}$$

Conjecture 3.7 ([CM18, Conjecture 1.5])

The Lagrangian capacity of the ellipsoid is

$$c_L(E(a_1, \ldots, a_n)) = \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1} = \delta_{\Omega(E(a_1,\ldots,a_n))}.$$

Conjecture 3.8 ([Per22, Conjecture 6.24])

If X_Ω is a convex or concave toric domain then

$$c_L(X_\Omega) = \delta_\Omega.$$

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To prove our results about the Conjecture 3.8, we will need to use the following symplectic capacities.

McDuff–Siegel capacities	$\tilde{\mathfrak{g}}_k^{\leq \ell}$	[MS22]
Higher symplectic capacities	$\mathfrak{g}_k^{\leq \ell}$	[Sie20]
Gutt–Hutchings capacities	c_k^{GH}	[GH18]

for $k, \ell \in \mathbb{Z}_{\geq 1}$. We will only need to consider these capacities for $\ell = 1$, i.e. $\tilde{\mathfrak{g}}_k^{\leq 1}, \mathfrak{g}_k^{\leq 1}$.

Theorem 4.1 ([Per22, Theorem 6.40])

If (X, λ) is a Liouville domain and $k \geq 1$ then $c_L(X) \leq \tilde{\mathfrak{g}}_k^{\leq 1}(X)/k$.

Proof sketch (1/2).

1. By definition of c_L, it suffices to assume that $L \subset X$ is an embedded Lagrangian torus and to prove that there exists a disk D with boundary on L with "small" symplectic area.
2. By definition of $\tilde{\mathfrak{g}}_k^{\leq 1}$, there exists a sequence u_t of J_t-holomorphic curves with bounded energy and satisfying a tangency constraint.
3. By the SFT compactness theorem, u_t converges to a broken holomorphic curve $(F_1, \ldots, F_N)$ (neck stretching along S^*L). Let C be the component of the limit which inherits the tangency constraint.

Proof sketch (2/2).

4. Since L is a torus, there is an upper bound on the Morse indices of the geodesics in L. Use this together with the SFT index formula to conclude that C must have at least $k+1$ punctures.
5. Therefore, C gives rise to k disks $D_1, \ldots, D_k$ in X with boundary in L. One of these disks is as desired. $\square$

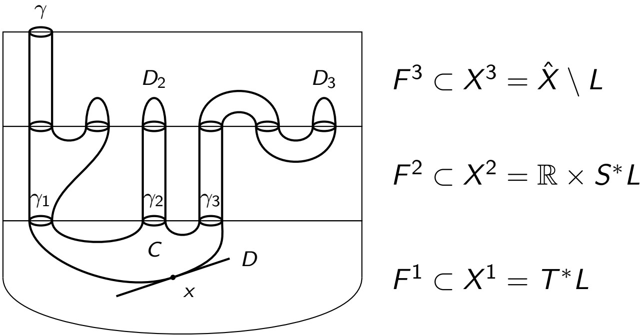

Figure: The broken holomorphic curve F in the case $N = 3$

Theorem 4.2 ([Per22, Theorem 6.41])

If X_Ω is a 4-dimensional convex toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

For every $k \in \mathbb{Z}_{\geq 1}$,

$$
\begin{aligned}
\delta_\Omega &\leq c_P(X_\Omega) && \text{[by Lemma 3.4]} \\
&\leq c_L(X_\Omega) && \text{[by Lemma 3.3]} \\
&\leq \tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega)/k && \text{[by Theorem 4.1]} \\
&= c_k^{\mathrm{GH}}(X_\Omega)/k && \text{[dim 4 and [MS22, Proposition 5.6.1]]} \\
&\leq c_k^{\mathrm{GH}}(N(\delta_\Omega))/k && \text{[X_Ω is convex, hence $X_\Omega \subset N(\delta_\Omega)$]} \\
&= \delta_\Omega(k+1)/k && \text{[by [GH18, Lemma 1.19]].}
\end{aligned}
$$

□

This finishes the proof of Conjecture 3.8 in the case where X_Ω is convex and 4-dimensional. We only used "convex + 4-dimensional" to say that $\tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega) = c_k^{\mathrm{GH}}(X_\Omega)$. This suggests the following conjecture.

Conjecture 4.3 ([Per22, Conjecture 6.42])

If X is a Liouville domain, $\pi_1(X) = 0$ and $c_1(TX)|_{\pi_2(X)} = 0$, then

$$c_L(X, \lambda) \leq \inf_k \frac{c_k^{\mathrm{GH}}(X, \lambda)}{k}.$$

We will now prove Conjecture 3.8 in full generality, but assuming that there exists a suitable virtual perturbation scheme which defines the curve counts of linearized contact homology. In this case, we can define Siegel's higher symplectic capacities $\mathfrak{g}_k^{\leq 1}$.

Theorem 4.4 ([Per22, Theorem 7.64])

If X is a Liouville domain such that $\pi_1(X) = 0$ and $2c_1(TX) = 0$ then $\mathfrak{g}_k^{\leq 1}(X) = c_k^{\mathrm{GH}}(X)$.

Proof sketch (1/2).

1. Let $E = E(a_1, \ldots, a_n)$ be a "skinny" ellipsoid such that there exists a strict exact symplectic embedding $\phi \colon E \longrightarrow X$. Consider the commutative diagram

$$\begin{tikzcd}
SH^{S^1,(\varepsilon,a]}_{n-1+2k}(X) \arrow[r, "\iota^{S^1,a}"] \arrow[d, hook, two heads, "\Phi^a_{\mathrm{BO}}"'] & SH^{S^1,+}_{n-1+2k}(X) \arrow[r, "\phi_!^{S^1}"] \arrow[d, hook, two heads, "\Phi_{\mathrm{BO}}"] & SH^{S^1,+}_{n-1+2k}(E) \arrow[d, hook, two heads, "\Phi_{\mathrm{BO}}"] \\
CH^a_{n-1+2k}(X) \arrow[r, "\iota^a"] \arrow[d, equal] & CH_{n-1+2k}(X) \arrow[r, "\phi_!"] \arrow[d, equal] & CH_{n-1+2k}(E) \arrow[d, "\epsilon_k^E"] \\
CH^a_{n-1+2k}(X) \arrow[r, "\iota^a"'] & CH_{n-1+2k}(X) \arrow[r, "\epsilon_k^X"'] & \mathbb{Q}
\end{tikzcd}$$

Proof sketch (2/2).

2. By definition of $\mathfrak{g}_k^{\leq 1}$ and c_k^{GH}, we only need to show that ϵ_k^E is an isomorphism, i.e. that the virtual count of curves (asymptotically cylindrical, satisfying a tangency constraint) in the ellipsoid is nonzero.
3. For this we show (using automatic transversality techniques) that moduli spaces of asymptotically cylindrical holomorphic curves in ellipsoid are transversely cut out.
4. Hence virtual counts agree with "usual" counts. We can count these curves explicitly (they are polynomials) and conclude that the count is nonzero. $\square$

Theorem 4.5 ([Per22, Theorem 7.65])

Assume that a suitable virtual perturbation scheme exists. If X_Ω is a convex or concave toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

$$
\begin{aligned}
\delta_\Omega &\leq c_P(X_\Omega) && \text{[by Lemma 3.4]} \\
&\leq c_L(X_\Omega) && \text{[by Lemma 3.3]} \\
&\leq \tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega)/k && \text{[by Theorem 4.1]} \\
&\leq \mathfrak{g}_k^{\leq 1}(X_\Omega)/k && \text{[by [MS22, Section 3.4]]} \\
&= c_k^{\mathrm{GH}}(X_\Omega)/k && \text{[by Theorem 4.4]} \\
&\leq c_k^{\mathrm{GH}}(N(\delta_\Omega))/k && \text{[}X_\Omega \text{ is convex, hence } X_\Omega \subset N(\delta_\Omega)\text{]} \\
&= \delta_\Omega(k+n-1)/k && \text{[by [GH18, Lemma 1.19]].}
\end{aligned}
$$

□

$$c_L(X) \leq \inf_k \frac{\tilde{\mathfrak{g}}_k^{\leq 1}(X)}{k}$$

$$\mathfrak{g}_k^{\leq 1}(X) = c_k^{\mathrm{GH}}(X)$$

$$c_L(X_\Omega) = \delta_\Omega$$

Thank you for listening!

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