Stops

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Definition (Liouville domain)

A **Liouville domain** is a pair (X, λ), where X is a compact, connected smooth manifold with boundary ∂X and $\lambda \in \Omega^1(X)$ is such that $\mathrm{d}\lambda \in \Omega^2(X)$ is symplectic, $\lambda|_{\partial X}$ is contact and the orientations on ∂X coming from $(X, \mathrm{d}\lambda)$ and coming from $\lambda|_{\partial X}$ are equal.

Definition (Liouville vector field)

If (X, λ) is a Liouville domain, it's **Liouville vector field** is the unique vector field Z such that $Z = \iota_Z \mathrm{d}\lambda$.

Lemma (Z is outward pointing)

If (X, λ) is a Liouville domain, then Z is outward pointing at ∂X.

Definition (symplectization)

Let (M, α) be a contact manifold. We define a new exact symplectic manifold, called the **symplectization** of (M, α), as follows. As a manifold, the symplectization is $\mathbb{R}^+ \times M$, with coordinate on $\mathbb{R}^+$ denoted by r. The symplectic potential of $\mathbb{R}^+ \times M$ is the 1-form $r\alpha$.

Lemma (a symplectic embedding for completions)

Let (X, λ) be a Liouville domain with Liouville vector field Z. Consider the "negative symplectization" $((0,1] \times \partial X, r\lambda|_{\partial X})$ of $(\partial X, \lambda|_{\partial X})$. Then, the flow of Z

$$\begin{aligned} \Phi_Z \colon (0,1] \times \partial X &\longrightarrow X \\ (t, x) &\longmapsto \phi_Z^{\ln t}(x) \end{aligned}$$

is an embedding such that $\Phi_Z^* \lambda = r\lambda|_{\partial X}$.

Definition (completion)

Let (X, λ) be a Liouville domain. We define an exact symplectic manifold $(\hat{X}, \hat{\lambda})$ called the **completion** of (X, λ), as follows. As a smooth manifold, $\hat{X}$ is the gluing of X and $\mathbb{R}^+ \times \partial X$ along

$$\Phi_Z \colon (0,1] \times \partial X \longrightarrow \Phi_Z((0,1] \times \partial X) \subset X$$

This gluing comes with smooth embeddings

$$\begin{aligned} \iota_X \colon X &\longrightarrow \hat{X}, \\ \iota_{\mathbb{R}^+ \times \partial X} \colon \mathbb{R}^+ \times \partial X &\longrightarrow \hat{X}. \end{aligned}$$

To define $\hat{\lambda}$, it suffices to say what is $\iota_X^* \hat{\lambda}$ and what is $\iota_{\mathbb{R}^+ \times \partial X}^* \hat{\lambda}$:

$$\begin{aligned} \iota_X^* \hat{\lambda} &\coloneqq \lambda, \\ \iota_{\mathbb{R}^+ \times \partial X}^* \hat{\lambda} &\coloneqq r \lambda|_{\partial X}. \end{aligned}$$

Definition (morphism of Liouville domains, [Syl19])

If (F, λ_F) and (M, λ_M) are Liouville domains, a **Liouville morphism** from F to M is a proper embedding $\phi\colon \hat{F} \longrightarrow \hat{M}$, such that

1. There exists $f : \hat{F} \longrightarrow \mathbb{R}$ compactly supported such that $\phi^*\hat{\lambda}_M = \hat{\lambda}_F + \mathrm{d}f$;
2. There exists a compact set K in $\hat{F}$ such that $\hat{Z}_F$ is ϕ-related to $\hat{Z}_M$ outside of K.

Definition (category of Liouville domains)

Liouville domains and morphisms of Liouville domains assemble into a category which we call **Liouv**.

- Identities: if $(M, \lambda) \in$ **Liouv**, then $\mathrm{id}_{(M,\lambda)} = \mathrm{id}_{\hat{M}}$.
- Composition: if $\phi \in \mathrm{Hom}(F, M)$ and $\psi \in \mathrm{Hom}(M, N)$, then $\psi * \phi \in \mathrm{Hom}(F, N)$ is given by $\psi * \phi = \psi \circ \phi\colon \hat{F} \longrightarrow \hat{M} \longrightarrow \hat{N}$.

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Lemma (Moser, [CE12, thm. 6.8], [Syl19, lem. 2.1])

Let (M, λ_t) be a family of Liouville domains, for $t \in [0,1]$ such that there exists $C \subset \operatorname{int} M$ compact for which $\lambda_t|_{M \setminus C} = \lambda_0|_{M \setminus C}$ for all t. Then, there exists a smooth family of Liouville isomorphisms $\phi_t \colon (M, \lambda_0) \longrightarrow (M, \lambda_t)$ such that ϕ_0 is the identity morphism.

Proof.

Consider $(\hat{M}, \hat{\lambda}_t)$. We will use **Moser's trick**. Define

- $X_t \in \mathfrak{X}(\hat{M})$ by $\dot{\hat{\lambda}}_t + \iota_{X_t} \mathrm{d}\hat{\lambda}_t = 0$
- $\phi_t \colon \hat{M} \longrightarrow \hat{M}$ to be the time dependent flow of X_t
- $f_t := \int_0^t \phi_s^* \iota_{X_s} \hat{\lambda}_s \mathrm{d}s$.

Compute $\frac{\mathrm{d}}{\mathrm{d}t} \phi_t^* \hat{\lambda}_t = \phi_t^*(\mathrm{d}\iota_{X_t} \hat{\lambda}_t)$ using Cartan's magic formula. Compute $\phi_t^* \hat{\lambda}_t = \hat{\lambda}_0 + \mathrm{d}f_t$ using the fundamental theorem of calculus. Then $\phi_t \colon \hat{M} \longrightarrow \hat{M}$ is the desired morphism $\phi_t \colon (M, \lambda_0) \longrightarrow (M, \lambda_t)$. □

Remark (deform Liouville form)

Let (X, λ) be a Liouville domain, $f\colon X \longrightarrow \mathbb{R}$ be a function with $\operatorname{supp} f \subset \operatorname{int} X$ and $\lambda' := \lambda + \mathrm{d}f$. Then, (X, λ') is a Liouville domain which is isomorphic to (X, λ) in **Liouv**.

Lemma (deform by changing collar)

Let (X, λ) be a Liouville domain, $f\colon \partial X \longrightarrow \mathbb{R}^+$ be a function and $X_f := \hat{X} \setminus \{(r, x) \in \mathbb{R}^+ \times \partial X \mid r > f(x)\}$, $\lambda_f = \hat{\lambda}|_{X_f}$. Then, (X_f, λ_f) is a Liouville domain which is canonically isomorphic to (X, λ) in **Liouv**.

Proof.

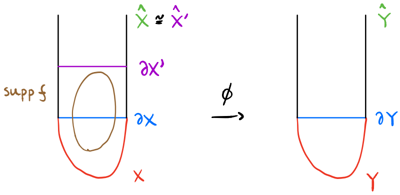
$$\begin{array}{ccc} X & \longrightarrow & \hat{X} \\ {\scriptstyle\iota}\downarrow & \nearrow & \downarrow{\scriptstyle\hat{\iota}} \\ X_f & \longrightarrow & \hat{X}_f \end{array}$$

Suffices to prove in the case $f(x) \geq 1$. Consider the inclusion $\iota\colon X \longrightarrow X_f$ and the induced map $\hat{\iota}\colon \hat{X} \longrightarrow \hat{X}_f$ (completion is functorial). Then, $\hat{\iota}\colon (\hat{X}, \hat{\lambda}_X) \longrightarrow (\hat{X}_f, \hat{\lambda}_f)$ is a diffeomorphism such that $\hat{\iota}^* \hat{\lambda}_f = \hat{\lambda}$. □

Lemma (from $\phi^*\hat{\lambda}_Y = \hat{\lambda}_X + \mathrm{d}f$ to $(\phi')^*\hat{\lambda}_Y = \hat{\lambda}'_X$)

Let $\phi\colon (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ be a morphism in **Liouv**. *Then there exist $\iota\colon (X, \lambda_X) \longrightarrow (X', \lambda'_X)$ an isomorphism and $\phi'\colon (X', \lambda'_X) \longrightarrow (Y, \lambda_Y)$ a morphism such that $(\phi')^*\hat{\lambda}_Y = \hat{\lambda}'_X$ and $\phi = \phi' * \iota$.*

Proof.



Choose $X' \subset \hat{X}$ a Liouville domain so big such that $\operatorname{supp} f \subset \operatorname{int} X'$. Define $\lambda'_X := \hat{\lambda}_X|_{X'} + \mathrm{d}f$. Notice that $(X, \lambda_X) \cong (X', \hat{\lambda}_X|_{X'}) \cong (X', \lambda'_X)$ and call the isomorphism $\iota\colon (X, \lambda_X) \longrightarrow (X', \lambda'_X)$. Define ϕ' so that $\phi' * \iota = \phi$. □

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We now consider **products** of Liouville domain. Our goal is to show that the operation of taking the product is a functor

$$\times : \mathbf{sk}(\mathbf{Liouv}) \times \mathbf{sk}(\mathbf{Liouv}) \longrightarrow \mathbf{sk}(\mathbf{Liouv}).$$

(Here $\mathbf{sk}(\mathbf{Liouv})$ denotes the **skeleton** of the category **Liouv**).

Product of Liouville domains is well defined up to iso.

If (M, λ_M), (N, λ_N) are Liouville domains, then $[M \times N, \lambda_M + \lambda_N] \in \mathbf{sk}(\mathbf{Liouv})$ is well defined.

Explanation

- $M \times N$ is a manifold with boundary and corners.
- We can smoothen the corners of $M \times N$ and get a Liouville domain. This procedure depends on a choice of smoothening.
- The completion of the smoothening is isomorphic (in **Liouv**) to $\hat{M} \times \hat{N}$, regardless of the choice of smoothening.

Induced maps on products are not quite Liouville morphisms

If (X, λ_X), (Y, λ_Y), (M, λ_M), (N, λ_N) are Liouville domains and

- $\phi\colon (X, \lambda_X) \longrightarrow (M, \lambda_M)$ is a morphism, $\phi^*\hat{\lambda}_M = \hat{\lambda}_X + \mathrm{d}f$;
- $\psi\colon (Y, \lambda_Y) \longrightarrow (N, \lambda_N)$ is a morphism, $\psi^*\hat{\lambda}_N = \hat{\lambda}_Y + \mathrm{d}g$.

Then $\phi \times \psi\colon \hat{X} \times \hat{Y} \longrightarrow \hat{M} \times \hat{N}$ is such that

$$(\phi \times \psi)^*(\hat{\lambda}_M \times \hat{\lambda}_N) = (\hat{\lambda}_X + \hat{\lambda}_Y) + \mathrm{d}(\underbrace{f \circ \phi + g \circ \psi}_{\text{not compactly supported}}).$$

Induced maps on products are well defined up to iso.

However, doing the trick from a previous lemma

- $\phi'\colon (X', \lambda'_X) \longrightarrow (M, \lambda_M)$ is a morphism, $(\phi')^*\hat{\lambda}_M = \hat{\lambda}'_X$;
- $\psi'\colon (Y', \lambda'_Y) \longrightarrow (N, \lambda_N)$ is a morphism, $(\psi')^*\hat{\lambda}_N = \hat{\lambda}'_Y$.

Then $\phi' \times \psi'\colon \hat{X}' \times \hat{Y}' \longrightarrow \hat{M} \times \hat{N}$ is such that

$$(\phi' \times \psi')^*(\hat{\lambda}_M \times \hat{\lambda}_N) = (\hat{\lambda}'_X + \hat{\lambda}'_Y).$$

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For $\rho > 0$, define $\mathbb{H}_\rho = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq -\rho\}$, equipped with $\lambda_{\mathbb{C}} = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x)$.

Definition (stop, [Syl19, def. 2.3])

A **stop** is a triple (M, F, σ), where (M, λ_M) is a $2n$-dimensional Liouville domain, (F, λ_F) is a $(2n-2)$-dimensional Liouville domain (the **fibre** of the stop), and $\sigma\colon \hat{F} \times \mathbb{H}_\rho \longrightarrow \hat{M}$ is a proper embedding for which there exists $f\colon \hat{F} \times \mathbb{H}_\rho \longrightarrow \mathbb{R}$ compactly supported such that $\sigma^*\hat{\lambda}_M = \hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}f$. The **width** of (M, F, σ) is ρ.

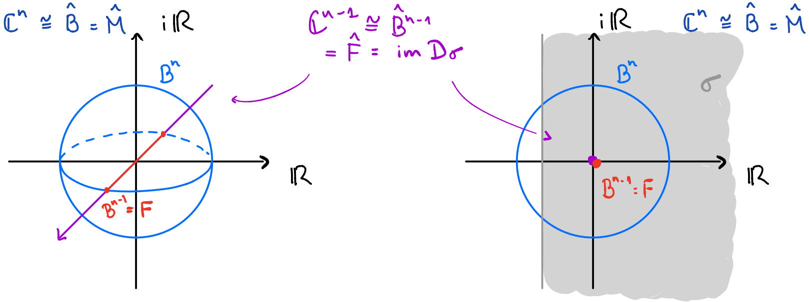
Definition (divisor, [Syl19, def. 2.3])

If (M, F, σ) is a stop, it's **divisor** is the map $D_\sigma := \sigma|_{\hat{F} \times \{0\}}\colon \hat{F} \longrightarrow \hat{M}$.

Example (stop)

We will give an example of a stop (M, F, σ).

- $(M, \lambda_M) := (B^n, \lambda_{\mathbb{C}^n})$, which implies $(\hat{M}, \hat{\lambda}_M) \cong (\mathbb{C}^n, \lambda_{\mathbb{C}^n})$;
- $(F, \lambda_F) := (B^{n-1}, \lambda_{\mathbb{C}^{n-1}})$, which implies $(\hat{F}, \hat{\lambda}_F) \cong (\mathbb{C}^{n-1}, \lambda_{\mathbb{C}^{n-1}})$;
- $\sigma \colon \mathbb{C}^{n-1} \times \mathbb{H}_\rho \longrightarrow \mathbb{C}^n$ is the inclusion map.



Lemma (divisors are Liouville morphisms)

If (M, F, σ) is a stop, its divisor is a Liouville morphism from F to M.

Proof.

Divisors preserve Liouville forms and Liouville vector fields:

$$\begin{aligned} D_\sigma^* \hat{\lambda}_M &= (\sigma \circ \iota_{\hat{F}})^* \hat{\lambda}_M && [\text{def. } D_\sigma] \\ &= \iota_{\hat{F}}^*(\hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}f) && [\sigma \text{ is a stop}] \\ &= \hat{\lambda}_F + \mathrm{d}(f \circ \iota_{\hat{F}}) && [\lambda_{\mathbb{C}}|_0 = 0]. \end{aligned}$$

$$\begin{aligned} &\mathrm{D}(D_\sigma)_x Z_{\hat{F}}|_x \\ &= \mathrm{D}\sigma|_{(x,0)} \circ \mathrm{D}\iota_{\hat{F}}|_x \cdot Z_{\hat{F}}|_x && [\text{def. } D_\sigma] \\ &= \mathrm{D}\sigma|_{(x,0)} \cdot Z_{\hat{F}\times\mathbb{H}_\rho}|_{(x,0)} && [\lambda_{\mathbb{C}}|_0 = 0] \\ &= Z_{\hat{M}}|_{D_\sigma(x)} && [\sigma^* \hat{\lambda}_M = \lambda_{\hat{F}\times\mathbb{H}_\rho} \text{ outside cpt.}]. \end{aligned}$$

□

For $\rho > 0$, $s \in (0, \frac{\pi}{2})$, define $A_s = \{re^{i\theta} \in \mathbb{C} \mid r > 0, |\theta| \leq s\}$ and $S_{\rho,s} = \overline{D}^2_\rho \cup A_s$, equipped with $\lambda_{\mathbb{C}} = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x)$.

Definition (narrow stop, [Syl19, def. 2.5])

A **narrow stop** is given by the data (M, F, σ), and is defined by replacing $\mathbb{H}_\rho$ by $S_{\rho,s}$ in the definition of a stop.

Remark (stops vs. narrow stops)

- Stops can be restricted to narrow stops.
- Strictly speaking, a narrow stop is not a stop.
- However, it's possible to construct a stop from a narrow stop.

Lemma (narrow stop $\rightsquigarrow$ stop, [Syl19, lem. 2.4])

Let $((M,\lambda_M),(F,\lambda_F),\sigma_0)$ be a narrow stop with parameters ρ, s. Then, there exist $\phi\colon \mathbb{H}_{\rho'} \longrightarrow S_{\rho,s}$ a proper embedding and $g\colon \hat{M} \longrightarrow \mathbb{R}$ such that if we define $\sigma := \sigma_0 \circ (\mathrm{id}\times\phi)$ then we have that

1. *The following diagram commutes:*

$$\begin{array}{ccc} \hat{F}\times\mathbb{R}_+ & \longrightarrow & \hat{F}\times\mathbb{H}_{\rho'} \\ \big\downarrow & \swarrow{\scriptstyle \mathrm{id}\times\phi} & \big\downarrow{\scriptstyle\sigma} \\ \hat{F}\times S_{\rho,s} & \xrightarrow[\sigma_0]{\qquad} & \hat{M} \end{array} \quad ;$$

2. $\operatorname{supp} g \subset \sigma(\hat{F}\times K)$, *for $K\subset\mathbb{H}_{\rho'}$ a compact neighbourhood of* 0*;*
3. $\sigma^*(\hat{\lambda}_M + \mathrm{d}g) = \hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}(f\circ(\mathrm{id}\times\phi))$.

Proof sketch.

The embedding ϕ is defined as $\phi = \phi_H^t$, for t big enough, where $H \colon \mathbb{C} \longrightarrow \mathbb{R}$ is a Hamiltonian which outside of a compact set is given by $r^2 \sin(\theta)$. Notice that the Hamiltonian flow of $r^2 \sin(\theta)$ preserves the Liouville form and it maps the real axis to the real axis. We choose t and ρ' so that $\phi = \phi_H^t$ maps $\mathbb{H}_{\rho'}$ to $S_{\rho,s}$. Then $\phi^* \lambda_{\mathbb{C}} = \lambda_{\mathbb{C}} - \mathrm{d}g'$, for $g' \colon \mathbb{H}_{\rho'} \longrightarrow \mathbb{R}$ with compact support K a neighbourhood of 0. Define $g \colon \hat{M} \longrightarrow \mathbb{R}$ so that $\sigma^* g = g'$ and $\operatorname{supp} g \subset \sigma(\hat{F} \times K)$.

$$\sigma \colon \hat{F} \times \mathbb{H}_{\rho'} \xrightarrow{\mathrm{id}_{\hat{F}} \times \phi} \hat{F} \times S_{\rho,\sigma} \xrightarrow{\sigma_0} \hat{M}$$

g'	$\sigma_0^* g$	g
$\hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}(f \circ (\mathrm{id}_{\hat{F}} \times \phi)) - \mathrm{d}g'$	$\hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}f$	$\hat{\lambda}_M$
$\hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}(f \circ (\mathrm{id}_{\hat{F}} \times \phi))$	$\hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}f + \mathrm{d}\sigma_0^* g$	$\hat{\lambda}'_M = \lambda_M + \mathrm{d}g$

□

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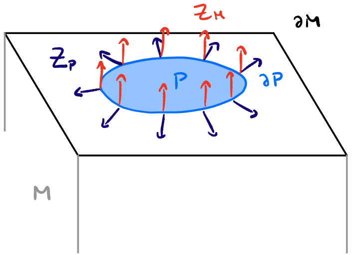
A **Liouville pair** is going to be defined analogously to the **Weinstein pairs** of [Eli17]. We will now study the relation between stops and Liouville pairs. This relation will be given by some special functions on Liouville pairs which we will call **functions with Liouville graph**.

Definition (Liouville pair, adapted from [Eli17, §1 and 2])

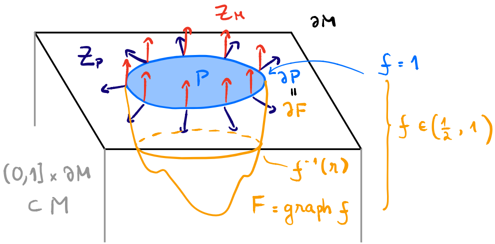
A **Liouville pair** is a tuple (M, P) such that (M, λ) is a Liouville domain of dimension $2n$ and $(P, \lambda|_P) \subset \partial M$ is a Liouville domain of dimension $2n-2$.



Definition (func. w. Liouville graph, adapted from [Syl19, prop. 2.6])

If (M, P) is a Liouville pair, a **function with Liouville graph** on (M, P) is a smooth function $f : P \longrightarrow [1/2, 1]$ such that

1. $f < 1$ on $\operatorname{int} P$;
2. $f|_{\partial P} = 1$;
3. All $r > 1/2$ are regular values and $f^{-1}(r) \subset P$ is contact;
4. $F := \operatorname{graph}(f) \subset M$ is a smooth submanifold such that $(Z_M)_p \in T_pF$ for all $p \in \partial P = \partial F$.



Proposition (Liouville pair defines stop, [Syl19, prop. 2.6])

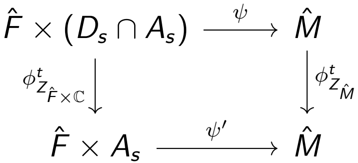
Let (M, P) be a Liouville pair and f be a function with Liouville graph on (M, P). Let $F = \operatorname{graph} f$. Then, $(F, \lambda|_F)$ is a Liouville domain and there exists $\sigma\colon \hat{F} \times \mathbb{H}_\rho \longrightarrow \hat{M}$ such that $\sigma^*(\hat{\lambda}_M + \mathrm{d}h) = \hat{\lambda}_F + \lambda_{\mathbb{C}} + \mathrm{d}f$ and $D_\sigma\colon \hat{F} \longrightarrow \hat{M}$ is induced from the inclusion $F \longrightarrow M$.

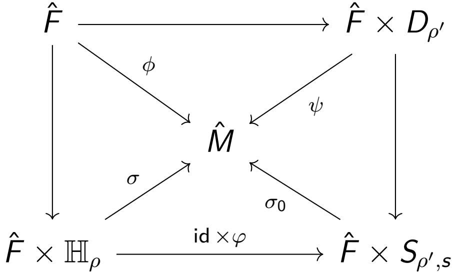
Proof sketch.

Show that $(F, \lambda|_F)$ is a Liouville domain. By functoriality of completions, $\iota\colon F \longrightarrow M$ defines $\phi\colon \hat{F} \longrightarrow \hat{M}$. It's possible to define an extension $\psi\colon \hat{F} \times D_{\rho'} \longrightarrow \hat{M}$ which is a symplectic embedding. Define $\theta := \lambda_{\hat{F} \times D_{\rho'}} - \psi^* \hat{\lambda}_M$, which is closed. Since $\theta|_{\hat{F}} = 0$, there exists an $h_0 \in C^\infty(D_{\rho'}, \mathbb{R})$ such that $\theta = \mathrm{d}h_0$. Define $h = \psi_* h_0$ and $\hat{\lambda}'_M = \hat{\lambda}_M + \mathrm{d}h$. Then, $\psi^* \hat{\lambda}'_M = \psi^* \hat{\lambda}_M + \mathrm{d}h_0 = \lambda_{\hat{F} \times D_{\rho'}}$.

Proof sketch (cont.)

Define $\psi' \colon \hat{F} \times A_s \longrightarrow \hat{M}$ as in the diagram. Patch together ψ and ψ' to get $\sigma_0 \colon \hat{F} \times S_{\rho',s} \longrightarrow \hat{M}$. Use the reasoning of a previous lemma to make an embedding $\varphi \colon \mathbb{H}_\rho \longrightarrow S_{\rho,s}$.





Define σ so that the diagram on the left commutes. Then, σ is as desired. □

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Thank you for listening!